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Optimal Time Scheduling for a Simple Reparable System

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Abstract— This work investigates an optimal control problem for scheduling repairs for a basic reparable system described by coupled partial and ordinary differential equations. Our goal is to maximize the system availability with optimal repair strategy. This problem leads to a bilinear optimal control structure in non-smooth space, posing significant computational challenges. We analyze the model, address the first-order necessary conditions for optimality, and develop a numerical algorithm to solve them. The efficacy of our approach is demonstrated through numerical experiments, which validate the theoretical results and illustrate the practical implementation of the control strategy.

I. Introduction

A reparable system operates under a maintenance strategy that initiates repair actions upon failure. The framework for modeling reparable multi-state systems by capturing failure and repair dynamics through state transitions between a functioning state and multiple failure modes was introduced by Chung in [5]. This model accounted for constant failure rates and general, arbitrarily distributed repair times, establishing a foundational structure for analyzing system availability over time. This work laid the foundation for future works of control reparable systems with arbitrarily distributed repair time governed by distributed parameter systems of coupled partial and ordinary hybrid equations (e.g., [7]–[9], [12], [13], [16], [17]). Much of the existing literature focused on the well-posedness and asymptotic behavior of the mathematical models.

In this paper, we address the practical control design for optimal time scheduling of a simple reparable system characterized by a single failure mode. Specifically, we allow the repair rate to vary with respect to the system running time. While the assumption of a single failure mode is a simplification, it captures the fundamental dynamics of the original model and serves to clearly demonstrate the core principles of our maintenance strategy. The simple reparable system reads:

$$\begin{aligned} \dot{p}_0(t) &= -\lambda p_0(t) + \int_0^L \mu(x,t)p_1(x,t)\,dx \\ \frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} &= -\mu(x,t)p_1(x,t), \end{aligned} \tag{1}$$

with boundary condition:

$$p_1(0,t) = \lambda p_0(t), \tag{2}$$

and initial conditions

$$p_0(0) = \phi_0, \quad p_1(x,0) = \phi_1(x), \tag{3}$$

where $0 \leq (\phi_0, \phi_1) \in \mathbb{R} \times L^1(0,L)$. Here:

1) $\lambda > 0$ is the constant failure rate of the device for failure mode;
2) $\mu(x,t) \geq 0$ is the space-time dependent nonnegative repair rate when the device is in failure state at time t with an elapsed repair time of x. It is assumed that for every $t \geq 0$,
$$\int_0^l \mu(x,t)\ dx < \infty, \quad \forall 0 \leq l < L,$$
$$\text{and} \quad \int_0^L \mu(x,t)\ dx = \infty;$$
3) $p_0(t)$ stands for the probability that the device is in good mode at time t;
4) $p_1(x,t)$ stands for the probability density (with respect to repair time) that the device is in failure mode at time t and has an elapsed repair time of x.

Optimal control design of reparable systems has been studied in (e.g. [4], [11]), which was mainly concerned with the time-independent repair rate $\mu(x,t) = \mu(x)$ as the control input using both L^1 and L^2-regularization for some specific initial data. Our work in [1] considered the bilinear controllability of μ that steers the system to any prescribed terminal state by a fixed time t_f from suitable initial states. In the subsequent work [2], we extended the design to a more sophisticated model including an intermediate degradable state and analyzed the resulting changes in controllability as the number of state modes increases.

This work designs a time-dependent repair function of the form

$$\mu(x,t) = g(t)\nu(x), \tag{4}$$

where $g \geq 0$ is a time-dependent control input and $\nu \geq 0$ is a prescribed spatial profile. The motivation for considering a separable space-time repair rate (4) arises from practical considerations in the design of repair

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policies. In many engineering systems such as industrial machinery, infrastructure networks, or medical devices (e.g., [3], [14], [15], [18]), the spatial repair profile ν (which encodes how repair efficiency depends on elapsed repair time) can be dictated by hardware limitations, aging behavior, or physical repair mechanisms, and thus cannot be easily controlled. For the sake of this limitation, the time-dependent part g captures the scheduling operating, which can desirably optimized in real time.

Under this structural assumption, we design the control repair through the following optimal control framework:

$$\min_{g\in U_{ad}} \quad J(g) = -\alpha \int_0^{t_f} p_0(t)dt - \beta p_0(t_f) + \gamma \|g\|_{L^1(0,t_f)} \tag{5}$$

subject to

$$\begin{aligned} \dot{p}_0(t) &= -\lambda p_0(t) + g(t)\int_0^L \nu(x)p_1(x,t)\,dx, \\ \frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} &= -g(t)\nu(x)p_1(x,t), \end{aligned} \tag{6}$$

with boundary and initial conditions (2)–(3) and the parameters $\alpha, \beta \geq 0$ and $\gamma > 0$, where α and β do not vanish simultaneously. Consider the set of admissible controls defined by

$$U_{ad} := \{g \in L^1(0,t_f) \,:\, 0 \leq g(t) \leq 1\}.$$

Throughout, we assume that the spatial profile ν is given and satisfies:

$$\begin{aligned} &\int_0^l \nu(x)\ dx < \infty, \quad \forall 0 \leq l < L, \\ \text{and} \quad &\int_0^L \nu(x)\ dx = \infty. \end{aligned} \tag{7}$$

In addition, we impose that

$$p_1(L,t) = 0. \tag{8}$$

The purpose of this condition is to guarantee conservation of the system regarding the probability distribution. A detailed discussion follows in the next section.

The optimization problem (5)–(8) is aimed at finding an optimal repair scheduling policy g that balances system availability (the probability the system is operational when needed) against repair efforts.

A. Properties of the Model

Before addressing the solution to the optimal control problem (5), it is necessary to understand the properties of the governing system and justify the feasibility of the problem. First of all, using the boundary condition (2) we have

$$\frac{dp_0}{dt} + \frac{d\int_0^L p_1(x,t)\,dx}{dt} = -p_1(L,t). \tag{9}$$

Applying the method of characteristics to solve the system (6) with the boundary and initial conditions (2)–(3) yields

$$p_1(x,t) = \begin{cases} \lambda p_0(t-x)e^{-\int_0^x g(s-x+t)\nu(s)ds}, & x < t, \\ \phi_1(x-t)e^{-\int_{x-t}^x g(s-x+t)\nu(s)ds}, & x \geq t, \end{cases} \tag{10}$$

and

$$p_0(t) = e^{-\lambda t}\phi_0 + \int_0^t e^{-\lambda(t-\tau)} \int_0^L \nu(x)p_1(x,\tau)\,dx\,g(\tau)\,d\tau. \tag{11}$$

For given $g(t) \geq 0$, it can be shown that the solution is nonnegative if the initial condition is nonnegative. Moreover, if $g(t) \equiv 1$, it is easy to check that (8) holds with the help of (10) and the assumption (7). In this case, the system governed by (6) and (2)–(3) is conserved in terms of probability distribution, that is,

$$p_0(t) + \int_0^L p_1(x,t)dx = \phi_0 + \int_0^L \phi_1(x)dx,$$

for every $t \geq 0$. A more detailed analysis regarding the well-posedness and stability of the time-independent case can be found in (e.g. [10], [13], [17]).

However, when $g(t)$ is time-dependent, assumption (7) on $\nu(x)$ alone is insufficient to ensure (8). Therefore, we impose this condition to guarantee the conservation of the system, which essentially poses an additional constraint on $g(t)$. On the other hand, it is clear that the constraint set of our current problem is non-empty by letting $g(t)$ be any positive constant.

Building on our previous study, this work addresses a more realistic scenario where repair operations need not be active continuously throughout the maintenance period. We model this using a non-smooth, L^1-based formulation for repair schedule. This approach naturally promotes temporal sparsity in the control. Consequently, our analysis requires non-smooth techniques. Furthermore, we extend this non-smooth analysis to a more complex setting: a non-negative and conservative ODE-PDE system, which introduces substantial challenges in both analysis and computation.

The paper is structured as follows: Section II presents the optimality conditions for problem (5). Section III introduces a numerical algorithm designed to meet these conditions, with its performance evaluated experimentally in Section IV. Finally, conclusions are drawn in Section V.

II. First-order Optimality Conditions

In this section, we derive the first-order optimality conditions for solving problem (5) subject to (6), (2)–(3), and (8). Due to page limit, the rigorous proofs concerning the existence of a solution, differentiability (subdifferentiability) of J and the regularity of the optimality system will be addressed elsewhere.

Let $\vec{q} = (q_0, q_1)^T$ satisfy

$$\frac{dq_0}{dt} = \lambda(q_0(t) - q_1(0,t)) - \alpha, \tag{12}$$

$$\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial x} = g(t)\nu(x)\left(q_1(x,t) - q_0(t)\right), \tag{13}$$

with the final time conditions

$$q_0(t_f) = \beta \quad \text{and} \quad q_1(x, t_f) = 0. \tag{14}$$

Theorem 2.1: For $0 \le (\phi_0, \phi_1) \in \mathbb{R}\times L^1(0,L)$, if g is an optimal solution to problem (5) and $\vec{p} = (p_0, p_1)^T$ is the corresponding solution to the state equations governed by (6), (2)–(3) together with (8), then there exists an adjoint state vector $\vec{q} = (q_0, q_1)^T$ satisfying (12)–(14), such that

$$g(t) = \begin{cases} 0, & \text{if} \quad H(t) > -\gamma, \\ \in [0,1], & \text{if} \quad H(t) = -\gamma, \\ 1, & \text{if} \quad H(t) < -\gamma, \end{cases} \tag{15}$$

for $t \in [0, t_f]$, where

$$H(t) := \int_0^L \nu(x) p_1(x,t)\left(q_1(x,t) - q_0(t)\right) dx. \tag{16}$$

Proof: We formally derive the first-order necessary optimality conditions using an Euler-Lagrangian approach. Define the Lagrangian by

$$\begin{aligned}
&\mathcal{L}(g, p_0, p_1; q_0, q_1, \zeta, \lambda_1, \lambda_2) := \\
&- \alpha \int_0^{t_f} p_0\, dt - \beta p_0(t_f) + \gamma \|g\|_{L^1(0,t_f)} \\
&+ \int_0^{t_f} \left(\dot{p}_0(t) + \lambda p_0(t) - g(t)\int_0^L \nu p_1\, dx\right) q_0\, dt \\
&+ \int_0^{t_f}\int_0^L \left(\frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial x} + g\nu p_1\right) q_1\, dx\, dt \\
&+ \int_0^{t_f} p_1(L,t)\zeta(t)\, dt \\
&- \int_0^{t_f} g\lambda_1\, dt + \int_0^{t_f} (g-1)\lambda_2\, dt,
\end{aligned}$$

where q_0 and q_1 are the Lagrange multipliers or the adjoint states corresponding to p_0 and p_1, respectively; ζ is the Lagrange multiplier introduced to handle the boundary constraint (8); and $\lambda_1 \ge 0$ and $\lambda_2 \ge 0$ are the KKT multipliers introduced to handle the inequality constrains imposed on U_{ad}.

Applying integration by parts together with the boundary conditions (2)–(8) and (9), we have

$$\begin{aligned}
&\mathcal{L}(g, p_0, p_1; q_0, q_1, \zeta, \lambda_1, \lambda_2) := \\
&- \alpha \int_0^{t_f} p_0(t)\, dt - \beta p_0(t_f) + \gamma \|g\|_{L^1(0,t_f)} \\
&+ p_0(t_f)q_0(t_f) - \phi_0 q_0(0) - \int_0^{t_f} p_0(t)\frac{dq_0}{dt}\, dt \\
&+ \lambda \int_0^{t_f} p_0(t)q_0(t)\, dt - \int_0^{t_f} \lambda p_0(t) q_1(0,t)\, dt \\
&+ \int_0^L p_1(x,t_f)q_1(x,t_f) - \phi_1(x)q_1(x,0)\, dx \\
&- \int_0^{t_f}\int_0^L g(t)q_0(t)\nu(x)p_1(x,t)\, dx\, dt \\
&+ \int_0^{t_f}\int_0^L g(t)\nu(x)q_1(x,t)p_1(x,t)\, dx\, dt \\
&- \int_0^{t_f}\int_0^L p_1(x,t)\left(\frac{\partial q_1}{\partial x} + \frac{\partial q_1}{\partial t}\right) \\
&+ p_0(0)\zeta(0) - p_0(t_f)\zeta(t_f) \\
&+ p_1(x,0)\zeta(0) - p_1(x,t_f)\zeta(t_f) \\
&+ \int_0^{t_f} p_0(t)\zeta'(t)\, dt + \int_0^{t_f}\left(\int_0^L p_1(x,t)\, dx\right)\zeta'(t)\, dt \\
&- \int_0^{t_f} g\lambda_1\, dt + \int_0^{t_f} (g-1)\lambda_2\, dt.
\end{aligned}$$

The adjoint states q_0 and q_1 are chosen such that the first derivatives of $\mathcal{L}$ with respect to p_0 and p_1 vanish, i.e. $\frac{\partial \mathcal{L}}{\partial p_0} = 0$ and $\frac{\partial \mathcal{L}}{\partial p_1} = 0$, which result in the adjoint equations

$$\frac{dq_0}{dt} = \lambda(q_0(t) - q_1(0,t)) + \zeta'(t) - \alpha, \tag{17}$$

$$\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial x} = g(t)\nu(x)\left(q_1(x,t) - q_0(t)\right) + \zeta'(t). \tag{18}$$

Letting $\frac{\partial \mathcal{L}}{\partial p_0(t_f)} = 0$ and $\frac{\partial \mathcal{L}}{\partial p_1(t_f)} = 0$ yields the final time conditions

$$q_0(t_f) = \beta + \zeta(t_f) \quad \text{and} \quad q_1(x,t_f) = \zeta(t_f). \tag{19}$$

Here we set

$$\zeta(0) = 0. \tag{20}$$

By Fermat principle (e.g., [6]), if g is an optimal solution, then

$$0 \in \gamma\partial\|g^*\|_{L^1(0,t_f)} + H(t) - \lambda_1(t) + \lambda_2(t), \tag{21}$$

where $H(t)$ is defined by (16). Recall that the subdifferential $\partial\|\cdot\|_{L^1(0,t_f)}$ at a point g consists of all the elements $\phi \in L^\infty(0,t_f)$ such that for $\forall h \in L^1(0,t_f)$,

$$\langle h - g, \phi\rangle_{L^1(0,t_f), L^\infty(0,t_f)} \le \|h\|_{L^1} - \|g\|_{L^1},$$

which is further defined as follows:

- if $g \neq 0$, then

$$\begin{aligned}\partial\|g\|_{L^1(0,t_f)} = \{\phi \in L^\infty(0,t_f) :\ & \|\phi\|_{L^\infty} = 1, \\ & \langle g, \phi\rangle_{L^1, L^\infty} = \|g\|_{L^1}\};\end{aligned}$$

- if $g=0$, then

$$\partial\|g\|_{L^1(0,t_f)} = \{\phi \in L^\infty(0,t_f) : \|\phi\|_{L^\infty} \le 1\}.$$

Let $\xi \in \partial\|g\|_{L^1(0,t_f)}$ be a subgradient of $\|\cdot\|_{L^1}$ at g. By (21), we have

$$H(t) = -\gamma\xi(t) + \lambda_1(t) - \lambda_2(t).$$

The complementary slack conditions are given by

$$\langle \lambda_1, g\rangle_{L^\infty,L^1} = \langle \lambda_2, 1-g\rangle_{L^\infty,L^1} = 0. \tag{22}$$

These conditions prevent the situation where $\lambda_1(t) > 0$ and $\lambda_2(t) > 0$ simultaneously. Then we consider three remaining situations.

- For $t \in [0,t_f]$ with $\lambda_1(t) > 0$ and $\lambda_2(t) = 0$: The complementary slackness conditions (22) imply $g(t) = 0$. Moreover, we have $\xi(t) \le 1$ and thus

  $$-\gamma\xi(t) + \lambda_1(t) - \lambda_2(t) > -\gamma.$$

  Hence $g(t) = 0$ if $H(t) > -\gamma$.
- For $t \in [0,t_f]$ with $\lambda_1(t) = 0$ and $\lambda_2(t) > 0$: The complementary slackness conditions (22) imply $g(t) = 1$. Moreover, since $t \in \operatorname{Supp} g(t)$, we have $\xi(t) = 1$ and thus

  $$-\gamma\xi(t) + \lambda_1(t) - \lambda_2(t) < -\gamma.$$

  As a result, $g(t) = 1$ if $H(t) < -\gamma$.
- For $t \in [0,t_f]$ with $\lambda_1(t) = 0$ and $\lambda_2(t) = 0$: The complementary slackness conditions (22) imply

  $$0 \le g(t) \le 1.$$

  When $g(t) \in (0,1]$, we have $\xi(t) = 1$ again, and thus

  $$-\gamma\xi(t) + \lambda_1(t) - \lambda_2(t) = -\gamma.$$

  Otherwise when $g(t) = 0$, we have

  $$-\gamma\xi(t) + \lambda_1(t) - \lambda_2(t) \ge -\gamma.$$

  Consequently, we have a replicated case of $H(t) > -\gamma$, and a new case that $g(t) \in [0,1]$ if $H(t) = -\gamma$.

Combining these three situations establishes the optimality condition (15).

Finally, observe that by defining $\tilde{q}_0(t) = q_0(t) - \zeta(t)$ and $\tilde{q}_1(x,t) = q_1(x,t) - \zeta(t)$, the pair $(\tilde{q}_0(t), \tilde{q}_1(x,t))$ satisfies the system (12)–(14) and the initial condition $(\tilde{q}_0(0), \tilde{q}_1(x,0)) = (q_0(0), q_1(x,0))$ by (20). Moreover, we have $q_1(x,t) - q_0(t) = \tilde{q}_1(x,t) - \tilde{q}_0(t)$. Thus H in (16) can be rewritten as

$$H(t) = \int_0^L \nu(x) p_1(x,t) \left(\tilde{q}_1(x,t) - \tilde{q}_0(t)\right) dx.$$

For notational simplicity, we retain $(q_0(t), q_1(x,t))$ to denote $(\tilde{q}_0(t), \tilde{q}_1(x,t))$ as the adjoint state vector, and this completes the proof. ■

III. Numerical Algorithm

The optimality condition (15) in Theorem 2.1 can be obtained via the fixed-point iteration

$$g^{k+1} = \mathcal{F}(g^k), \tag{23}$$

where the mapping $\mathcal{F}$ encodes the system given by the state equation (6), the adjoint equation (12)–(14), and optimality condition (15). The fixed-point iteration (23) is numerically implemented according to the following computational sequence: Starting with an initial g^0, we iteratively update

$$g^k \to (p_0^{k+1}, p_1^{k+1}) \to (q_0^{k+1}, q_1^{k+1}) \to g^{k+1}$$

until some prescribed stopping criteria are satisfied, where (p_0^{k+1}, p_1^{k+1}) solves the state equation (6) with $g = g^k$, (q_0^{k+1}, q_1^{k+1}) solves the adjoint equation (12)–(14) with $g = g^k$. In particular for updating g^{k+1}, we first denote by g_{temp}^{k+1} the control obtained via the optimality condition (15) with (p_0^{k+1}, p_1^{k+1}) and (q_0^{k+1}, q_1^{k+1}), and we introduce a relaxation factor $\theta^{k+1} \in (0,1]$ to control the update:

$$g^{k+1} = (1-\theta^{k+1})g^k + \theta^{k+1} g_{\text{temp}}^{k+1}. \tag{24}$$

This relaxation in our fixed-point iteration enhances stability by preventing overshooting and oscillations between iterations. For the numerical implementation of the second case in the optimality condition (15), we define $g^*(t)$ as a uniformly distributed random variable in the interval $[0,1]$, generated using the MATLAB command rand. For state and adjoint equations, we discretize both the temporal and spatial dimensions via finite difference methods. The implementation strategy is outlined in Algorithm 1.

Algorithm 1: Fixed-Point Iteration for the Optimal Time Scheduling Problem

Require: Initial guess g^0; parameters $\alpha, \beta, \gamma > 0$; tolerance $\epsilon > 0$
1: repeat
2: Solve the state equation (6) with boundary and initial conditions (2)–(3) and (8) using $g = g^k$
3: Solve the adjoint equation (12)–(14) with $g = g^k$
4: Compute intermediate control g_{temp}^{k+1} from the optimality condition (15) using the new states (p_0^{k+1}, p_1^{k+1})
5: Update g^{k+1} via the relaxation scheme (24) with a factor in $(0,1]$
6: $k \leftarrow k+1$
7: until $\min(|J(g^k) - J(g^{k-1})|/|J(g^{k-1})|, |J(g^k) - J(g^{k-1})|) < \epsilon$

IV. Numerical Experiments

To validate the proposed framework, we conduct numerical experiments assessing the performance of the optimal scheduling strategy.

We consider a domain of length $L = 1$, final time $t_f = 1$, and initial data $\phi_0(x) = 1$ and $\phi_1(x) = 0$. Let the constant system failure rate be $\lambda = 0.5$ and the spatial profile be $\nu(x) = (L - x)^{-1}$. We adopt a uniform spatial-temporal grid with $\Delta x = \Delta t = 0.05$.

The convergence threshold is set to $\epsilon = 10^{-5}$ for optimal control updates, i.e., the g-iteration terminates once either the absolute or relative differences of the cost functional values between successive iterations become smaller than ϵ. Additionally, we implement a safeguard stopping criterion that terminates the fixed-point iteration if the iteration count exceeds 100. The relaxation parameter θ^k follows a piecewise-constant schedule:

$$\theta^k = \begin{cases} 1, & \text{if } k \leq 10, \\ 0.5, & \text{if } 10 < k \leq 20, \\ 0.1, & \text{if } 20 < k \leq 30, \\ 0.01, & \text{if } k > 30, \end{cases}$$

where k denotes the iteration count.

Next we conduct our numerical experiments on optimal time scheduling. Setting $\gamma = 0.05$, we examine three parameter combinations:

$$(\alpha, \beta) = (1, 0),\ (0, 1),\ \text{and } (1, 1).$$

The corresponding optimal controls and solutions to the state and adjoint equations are presented in Figures 1, 2, and 3, respectively. These numerical results demonstrate the significant impact of parameter selection on the optimal control strategy for the repairable system.

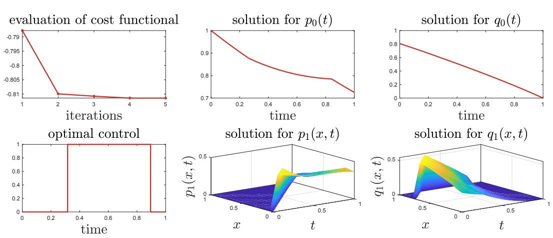

Fig. 1. Optimal time scheduling with $(\alpha, \beta, \gamma) = (1, 0, 0.05)$

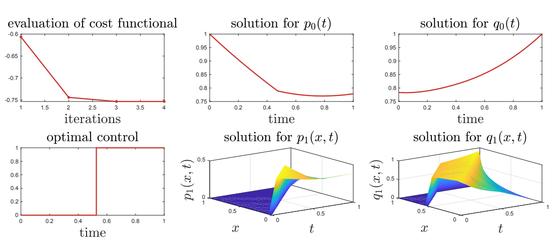

Fig. 2. Optimal time scheduling with $(\alpha, \beta, \gamma) = (0, 1, 0.05)$

We then present two sets of comprehensive phase transition analysis involving system failure rate λ and regularization parameter α, β, and γ for our control design.

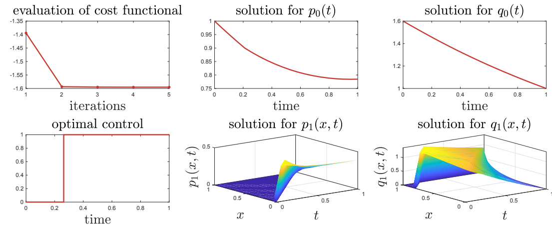

Fig. 3. Optimal time scheduling with $(\alpha, \beta, \gamma) = (1, 1, 0.05)$

- With fixed parameters
$$\gamma = 0.1 \text{ and } \lambda = 0.25,$$
we examine how the overall performance $\int_0^{t_f} p_0(t)dt$ and terminal state $p_0(t_f)$ vary across different (α, β) combinations. The parameters α and β are varied from 0 to 3 and 4, respectively, in increments of 0.25. Figure 4 illustrates the resulting phase transition diagram. As α and β increase, we observe a significant improvement in the overall performance of the reparable system in good mode. Moreover, the system exhibits distinct sensitivity regimes depending on the parameter values: for small α, the terminal state $p_0(t_f)$ shows strong dependence on β, indicating a sensitive regime, whereas for larger α, it becomes relatively insensitive to β variations.
- With fixed parameters
$$\alpha = \beta = 2.5,$$
we additionally examine how values of $\int_0^{t_f} p_0(t)dt$ and $p_0(t_f)$ vary with different constant system failure rate λ (taking discrete values from 0.1 to 0.85 in increments of 0.0625) and the regularization parameter γ (ranging from 0 to 1 in increments of 0.0625). Figure 5 illustrates the observed phase transition phenomenon. Our results demonstrate systematic performance degradation with increasing values of both the system failure rate λ and the regularization parameter γ. While the negative correlation with λ is expected (as higher failure rates naturally impair system performance), the γ-dependence reveals a more subtle control-theoretic mechanism: larger γ values in the cost functional increasingly constrain the effort of repairing, driving it toward zero and consequently reducing system capability.

V. Conclusion

This paper has introduced a novel framework for the optimal time scheduling of a reparable system governed by coupled ODE–PDEs. To manage the complexity of optimizing a space-time dependent repair rate $\mu(x, t)$, we proposed a separable structure $\mu(x, t) = g(t)\nu(x)$, where $\nu(x)$ is a prescribed spatial repair profile, and we aim to optimize the time-dependent function $g(t)$ within $L^1(0, t_f)$ space. This formulation simplifies the control design while maintaining practical applicability.

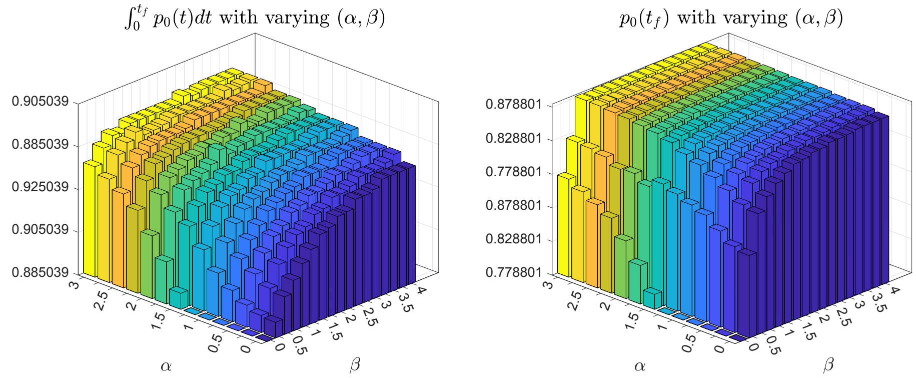

Fig. 4. Overall performance $\int_0^{t_f} p(t)dt$ and terminal state $p(t_f)$ with various (α, β) combinations and fixed $(\lambda, \gamma) = (0.25, 0.1)$.

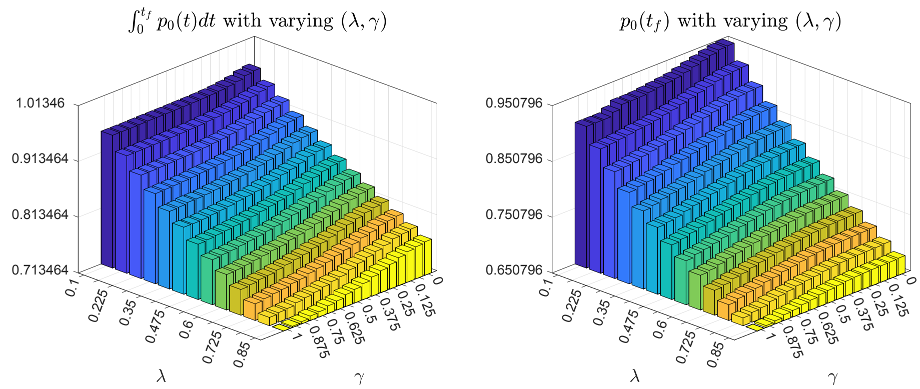

Fig. 5. Overall performance $\int_0^{t_f} p(t)dt$ and terminal state $p(t_f)$ with various (λ, γ) combinations and fixed $(\alpha, \beta) = (2.5, 2.5)$.

The approach enhances computational tractability and naturally promotes sparsity (i.e., intermittent) repair schedules. The use of a non-smooth cost functional and time-dependent control generalizes prior results. The case concerning the general space-time repair strategies will be addressed in our future work.

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