SIAM J. NUMER. ANAL.
Vol. 64, No. 1, pp. 148–169




# SPHERICAL CONFIGURATIONS AND QUADRATURE METHODS FOR INTEGRAL EQUATIONS OF THE SECOND KIND*

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**Abstract.** In this paper, we propose and analyze a product integration method for the second-kind integral equation with weakly singular and continuous kernels on the unit sphere $\mathbb{S}^2$. We employ quadrature rules that satisfy the Marcinkiewicz–Zygmund property to construct hyperinterpolation for approximating the product of the continuous kernel and the solution in terms of spherical harmonics. By leveraging this property, we significantly expand the family of candidate quadrature rules and establish a connection between the geometrical information of the quadrature points and the error analysis of the method. We then utilize product integral rules to evaluate the singular integral, with the integrand being the product of the singular kernel and each spherical harmonic. We derive a practical $L^\infty$ error bound, which consists of two terms: one controlled by the best approximation of the product of the continuous kernel and the solution and the other characterized by the Marcinkiewicz–Zygmund property and the best approximation polynomial of this product. Numerical examples validate our numerical analysis.

**Key words.** integral equation, product integration, Nyström method, quadrature, hyperinterpolation, point distribution

**MSC codes.** 45B05, 65R20, 65D32, 41A10

**DOI.** 10.1137/24M1688370

## 1. Introduction.

We consider the Fredholm integral equation of the second kind

$$\varphi(\boldsymbol{x}) - \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})d\omega(\boldsymbol{y}) = f(\boldsymbol{x}) \tag{1.1}$$

on the sphere $\mathbb{S}^2 := \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\|_2 = 1\}$ with surface measure $d\omega$, where $|\boldsymbol{x}-\boldsymbol{y}| := [2(1-\boldsymbol{x}\cdot\boldsymbol{y})]^{1/2}$ denotes the Euclidean distance between points $\boldsymbol{x}$ and $\boldsymbol{y}$ on $\mathbb{S}^2$. The inhomogeneous term $f$ and the kernel $K$ are assumed to be continuous on $\mathbb{S}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$, respectively, whereas the weight function $h : (0,\infty) \to \mathbb{R}$ is allowed to be weakly singular. Consequently, the integral operator in (1.1) is compact. Throughout this paper, it is also assumed that the corresponding homogeneous equation to (1.1) has no nontrivial solutions. Therefore, by the classic Riesz theory [32], the inhomogeneous equation (1.1) has a unique solution $\varphi$ that continuously depends on $f$, ensuring the continuity of $\varphi$. In the past decades, numerical methods for (1.1) have attracted much attention, and many important works have emerged; see, for example, [10, 13, 14, 15, 18, 21, 30, 36] and references therein.

In this paper, we are concerned with a class of quadrature-type methods for the numerical solution of (1.1). More precisely, we are interested in computing the approximate solution $\varphi_n$ that satisfies an equation of the form

*Received by the editors August 28, 2024; accepted for publication (in revised form) October 21, 2025; published electronically February 2, 2026.
https://doi.org/10.1137/24M1688370
**Funding:** This work was partially supported by the National Natural Science Foundation of China (no. 12371099) and the Special Posts of Guizhou University (no. [2024]42).
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$$\varphi_n(\boldsymbol{x})-\sum_{j=1}^{m} W_j(\boldsymbol{x})K(\boldsymbol{x},\boldsymbol{x}_j)\,\varphi_n(\boldsymbol{x}_j)=f(\boldsymbol{x}),\quad \boldsymbol{x}\in\mathbb{S}^2, \tag{1.2}$$

where $\varphi_n$ is the computed solution with index $n$ explained in section 2, $\mathcal{X}_m := \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m\}\subset\mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$, and $W_1,\ldots,W_m$ are continuous scalar-valued functions on $\mathbb{S}^2$. For a fixed value of $\boldsymbol{x}$, we may interpret $\{\boldsymbol{x}_j\}$ and $\{W_j\}$ as the points and weights in a certain quadrature rule

$$\sum_{j=1}^{m} W_j(\boldsymbol{x})g(\boldsymbol{x}_j)\approx\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)g(\boldsymbol{y})d\omega(\boldsymbol{y}) \tag{1.3}$$

for some continuous function $g$, in which the possibly singular kernel $h(|\boldsymbol{x}-\boldsymbol{y}|)$ has been incorporated implicitly into the weights $W_j(\boldsymbol{x})$. In particular, if $h(|\boldsymbol{x}-\boldsymbol{y}|)\equiv 1$, then weights $W_j$ become constants $w_j$, and the quadrature (1.3) reduces to the classic quadrature form with scalar positive weights $w_j$, as follows:

$$\sum_{j=1}^{m} w_j g(\boldsymbol{x}_j)\approx\int_{\mathbb{S}^2} g(\boldsymbol{y})d\omega(\boldsymbol{y}). \tag{1.4}$$

The goal of this paper is to propose an intrinsic scheme for the integral equation (1.1) on $\mathbb{S}^2$ by providing an explicit construction of $W_j(\boldsymbol{x})$ and to establish a unified convergence theory for approximations of the form (1.2), incorporating with the geometry (distribution information) of $\mathcal{X}_m\subset\mathbb{S}^2$. This convergence theory not only addresses convergence itself, which can be achieved using Anselone's collectively compact operator approximation theory [6] with Sloan's modification for the singular term [33] but also the *rate* of convergence, derived with the aid of the authors' recent work on hyperinterpolation [3, 4, 5]. It should also be noted that Sloan's modification in [33] for singular kernels was proposed for equations on the interval, while we are extending it to the sphere $\mathbb{S}^2$ in this paper.

In traditional numerical analysis, the chosen points $\boldsymbol{x}_j$ are often required to satisfy some exactness assumptions, that is, the quadrature rule in these points is assumed to be exact for polynomials of certain degrees. A typical choice of $\mathcal{X}_m$ on spheres is spherical $t$-designs [17]. However, in more realistic applications, the available points may not fulfill this desired assumption due to economic or engineering limitations. To address this issue, we assume $\mathcal{X}_m$ to satisfy the Marcinkiewicz–Zygmund property without exactness assumption in this paper. Consequently, besides spherical $t$-designs, many spherical configurations are feasible for constructing the quadrature rule (1.4) and the weights $W_j(\boldsymbol{x})$, such as randomly distributed points [23], equal area points [31], minimal energy points [37], maximal determinant points (Fekete points) [37], and more. Even for a set of scattered points without any special structure or properties, we can extract their geometrical information, such as the mesh norm, to satisfy the MZ property for polynomials of certain degrees (see, for example, [20, 27]). Therefore, the family of spherical configurations for the approximation form (1.2) in solving the integral equation (1.1) can be significantly enriched.

In section 2, we propose our scheme and justify the underlying assumptions. Section 3 establishes the convergence theory. Section 4 discusses three specific singular kernels and addresses techniques for computing with them. Numerical experiments are presented in section 5. Finally, section 6 provides concluding discussions.

**2. Proposed scheme.** We denote by $\{Y_{\ell,k}: k=1,\ldots,2\ell+1,\ \ell=0,1,2,\ldots\}$ the $L^2$-orthogonal system of real spherical harmonics of degree $\ell$ and order $k$ [29] and let $\mathbb{P}_n$ be the polynomial space of degree $n$ that is spanned by spherical harmonics of degree at most $n$. Let $L^p(\mathbb{S}^2)$ be the usual $L^p$ space on $\mathbb{S}^2$ with $1\leq p\leq\infty$, and let $C(\mathbb{S}^2)$ be the space of continuous functions on $\mathbb{S}^2$ equipped with the uniform norm $\|g\|_{L^\infty}:=\max_{\boldsymbol{x}\in\mathbb{S}^2}|g(\boldsymbol{x})|$.

**2.1. Hyperinterpolation.** For a continuous function $g\in C(\mathbb{S}^2)$, its hyperinterpolation of degree $n$ is defined as

$$\mathcal{L}_n g := \sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\langle g, Y_{\ell,k}\rangle_m Y_{\ell,k}, \tag{2.1}$$

where the coefficient

$$\langle g, Y_{\ell,k}\rangle_m := \sum_{j=1}^{m} w_j g(\boldsymbol{x}_j)Y_{\ell,k}(\boldsymbol{x}_j)$$

is numerically evaluated by the quadrature rule (1.4) with quadrature points $\mathcal{X}_m$ and positive quadrature weights $w_j>0$, $j=1,\ldots,m$. The approximation scheme is due to Sloan [34], in which the quadrature rule (1.4) is assumed to be exact for polynomials (spherical harmonics) of degree at most $2n$, that is,

$$\sum_{j=1}^{m} w_j\chi(\boldsymbol{x}_j)=\int_{\mathbb{S}^2}\chi(\boldsymbol{x})d\omega(\boldsymbol{x}) \quad \forall\chi\in\mathbb{P}_{2n}. \tag{2.2}$$

Due to the highly restrictive nature of quadrature exactness, obtaining data that ensure the corresponding quadrature meets the desired exactness is often impractical or impossible. In a series of recent works [3, 4, 5] by the authors, it was shown that this exactness assumption can be relaxed to an assumption inspired by the Marcinkiewicz–Zygmund inequality [16, 20, 23, 25, 26, 27, 28]. Let $\delta_{\mathcal{X}_m}$ be the mesh norm of $\mathcal{X}_m\subset\mathbb{S}^2$ defined as

$$\delta_{\mathcal{X}_m} := \max_{\boldsymbol{x}\in\mathbb{S}^2}\min_{\boldsymbol{x}_j\in\mathcal{X}_m}\arccos(\boldsymbol{x}\cdot\boldsymbol{x}_j),$$

which can be regarded as the geodesic radius of the largest hole in the mesh $\mathcal{X}_m$. It was investigated in [20, 27] that for every $\eta\in(0,1)$ and every polynomial $\chi\in\mathbb{P}_n$, the Marcinkiewicz–Zygmund inequality

$$(1-\eta)\int_{\mathbb{S}^2}\chi^2 d\omega \leq \sum_{j=1}^{m} w_j\chi(\boldsymbol{x}_j)^2 \leq (1+\eta)\int_{\mathbb{S}^2}\chi^2 d\omega \quad \forall\chi\in\mathbb{P}_n \tag{2.3}$$

holds if

$$n\lesssim \eta/\delta_{\mathcal{X}_m}. \tag{2.4}$$

The mesh norm represents the covering radius, defined as the smallest radius for $m$ identical spherical caps centered at points $\boldsymbol{x}_j$ that fully cover the sphere. Then, the whole area of all the caps must be at least that of the sphere, leading to

$$m2\pi\left(1-\cos(\delta_{\mathcal{X}_m})\right)=m4\pi\sin^2(\delta_{\mathcal{X}_m}/2)\geq 4\pi.$$

Using the inequality $\sin x \le x$ for $x \in [0,\pi]$, we have a lower bound $\delta_{\mathcal{X}_m} \ge 2m^{-1/2}$ on the mesh norm; see, e.g., [2]. The condition (2.4) ensuring the Marcinkiewicz–Zygmund inequality (2.3) thus implies the bound

$$m \gtrsim (n/\eta)^2 . \tag{2.5}$$

Moreover, this inequality (2.3) even holds when $\mathcal{X}_m$ consists of random points. When the quadrature rule (1.4) is equal-weight, it was shown in [23] that if an independent random sample of $m$ points drawn from the measure $d\omega$, then there exists a constant $\bar{c} := \bar{c}(\gamma)$ such that for every $\eta \in (0,1)$, the MZ inequality (2.3) holds with probability exceeding $1 - \bar{c}n^{-\gamma}$ on the condition of

$$m \ge \bar{c}n^2 \log n/\eta^2. \tag{2.6}$$

These results imply that for any $n \in \mathbb{N}$ and any $\eta$ (even when $\eta$ depends on $n$), the Marcinkiewicz–Zygmund inequality (2.3) holds provided the number $m$ of quadrature points is sufficiently large, as specified by (2.5) or (2.6). This fact is a crucial component of the assumptions for our numerical scheme (1.2); see section 2.3.3.

In our previous work [4] on hyperinterpolation, if the $m$-point positive-weight quadrature rule (1.4) satisfies the Marcinkiewicz–Zygmund inequality (2.3) of degree $n$ with $\eta \in [0,1)$, then

$$\|\mathcal{L}_n g\|_{L^2} \le \sqrt{1+\eta}\left(\sum_{j=1}^{m} w_j\right)^{1/2} \|g\|_{L^\infty} \tag{2.7}$$

and

$$\|\mathcal{L}_n g - g\|_{L^2} \le \left(\sqrt{1+\eta}\left(\sum_{j=1}^{m} w_j\right)^{1/2} + |\mathbb{S}^2|^{1/2}\right) E_n(g) + \sqrt{\eta^2+4\eta}\|\chi^*\|_{L^2}, \tag{2.8}$$

where $E_n(g) := \inf_{\chi\in\mathbb{P}_n(\mathbb{S}^2)} \|g-\chi\|_{L^\infty}$ denotes the best uniform error of $g$ by a polynomial in $\mathbb{P}_n(\mathbb{S}^2)$, and $\chi^* \in \mathbb{P}_n(\mathbb{S}^2)$ denotes the best approximation polynomial of $g$ in $\mathbb{P}_n(\mathbb{S}^2)$ in the sense of $\|g-\chi^*\|_{L^\infty} = E_n(g)$. Hence, the construction of hyperinterpolation is feasible with many more quadrature rules outside the traditional candidates.

**2.2. Numerical scheme.** For the integral equation (1.1), we propose to replace the term $K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})$, as a function of $y \in \mathbb{S}^2$ with its *hyperinterpolation* of degree $n$. Therefore, for the integral operator in (1.1), we have its approximation

$$\begin{aligned}
&\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)\mathcal{L}_n\left(K(\boldsymbol{x},\cdot)\varphi\right)(\boldsymbol{y})d\omega(\boldsymbol{y})\\
&\quad= \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)\left(\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1} \langle K(\boldsymbol{x},\cdot)\varphi, Y_{\ell,k}\rangle_m Y_{\ell,k}(\boldsymbol{y})\right)d\omega(\boldsymbol{y})\\
&\quad= \sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\left(\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y})\right)\langle K(\boldsymbol{x},\cdot)\varphi, Y_{\ell,k}\rangle_m\\
&\quad= \sum_{j=1}^{m} w_j\left(\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\left(\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y})\right)Y_{\ell,k}(\boldsymbol{x}_j)\right)K(\boldsymbol{x},\boldsymbol{x}_j)\varphi(\boldsymbol{x}_j)\\
&\quad=: \sum_{j=1}^{m} W_j(\boldsymbol{x})K(\boldsymbol{x},\boldsymbol{x}_j)\varphi(\boldsymbol{x}_j),
\end{aligned} \tag{2.9}$$

where

$$W_j(\boldsymbol{x}) := w_j \left( \sum_{\ell=0}^{n} \sum_{k=1}^{2\ell+1} \left( \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|) Y_{\ell,k}(\boldsymbol{y}) d\omega(\boldsymbol{y}) \right) Y_{\ell,k}(\boldsymbol{x}_j) \right). \tag{2.10}$$

One of the key theoretical contributions of this paper establishes the existence of the numerical solution $\varphi_n$, obtained via the Nyström scheme (1.2) with weights (2.10) under specific conditions. In particular, we require that the quadrature rule (1.4), used to construct $\varphi_n$, has a mesh norm $\delta_m$ satisfying condition (2.4), thereby ensuring the Marcinkiewicz–Zygmund property (2.3) of degree $n$. Here, the constant $\eta$ occurred in the property (2.3) depends on $n$. Another key contribution is the derivation of an error bound for the numerical scheme (1.2), valid under the assumption that the numerical solution exists. For a complete discussion, we refer to Theorem 3.1.

Numerically, the solution $\varphi_n(\boldsymbol{x})$ at an arbitrary point $\boldsymbol{x} \in \mathbb{S}^2$ can be evaluated in two stages. First, one numerically solves the system of linear equations

$$\varphi_i^{(n)} - \sum_{j=1}^{m} W_j(\boldsymbol{x}_i) K(\boldsymbol{x}_i, \boldsymbol{x}_j) \varphi_j^{(n)} = f(\boldsymbol{x}_i), \quad i = 1, \ldots, m \tag{2.11}$$

for the quantities $\varphi_j^{(n)}$, $j = 1, \ldots, m$. Second, the value of $\varphi_n(\boldsymbol{x})$ at any $\boldsymbol{x} \in \mathbb{S}^2$ can be evaluated by the direct usage of (1.2),

$$\varphi_n(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_{j=1}^{m} W_j(\boldsymbol{x}) K(\boldsymbol{x}, \boldsymbol{x}_j) \varphi_j^{(n)}, \quad \boldsymbol{x} \in \mathbb{S}^2. \tag{2.12}$$

The solvability of the two-stage scheme (2.11) and (2.12) is characterized as follows.

THEOREM 2.1 (Solvability of the two-stage scheme). *The following two statements hold:*

- *Let $\varphi_n$ be the numerical solution computed by the Nyström scheme* (1.2) *with weights* (2.10). *Then, the system* (2.11) *of linear equations is solvable with solutions $\varphi_j^{(n)} = \varphi_n(\boldsymbol{x}_j)$, $j = 1, \ldots, m$.*
- *Let $\varphi_j^{(n)}$, $j = 1, \ldots, m$, be a solution of the system* (2.11). *Then, the function defined by* (2.12) *is a solution of the Nyström scheme* (1.2).

*Proof.* The first statement is trivial; we let $\boldsymbol{x} = \boldsymbol{x}_j$, $j = 1, \ldots, m$, in the Nyström scheme (1.2). For a solution $\varphi_j^{(n)}$, $j = 1, \ldots, m$, of the system (2.11), the function $\varphi_n$ defined by (2.12) takes values

$$\varphi_n(\boldsymbol{x}_i) = f(\boldsymbol{x}_i) + \sum_{j=1}^{m} W_j(\boldsymbol{x}_i) K(\boldsymbol{x}_i, \boldsymbol{x}_j) \varphi_j^{(n)} = \varphi_i^{(n)}, \quad i = 1, \ldots, m,$$

where the second equality is due to (2.11). Inserting $\varphi_j^{(n)} = \varphi_n(\boldsymbol{x}_j)$, $j = 1, \ldots, m$, into the function $\varphi_n$ defined by (2.12), we show that it solves the Nyström scheme (1.2). ∎

**2.3. Assumptions and their justification.** Let the integral operator in (1.1) be denoted by

$$(A\varphi)(\boldsymbol{x}) := \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|) K(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) d\omega(\boldsymbol{y}), \quad \varphi \in C(\mathbb{S}^2), \quad \boldsymbol{x} \in \mathbb{S}^2.$$

We also denote by

$$(A_n\varphi)(\boldsymbol{x}) := \sum_{j=1}^{m} W_j(\boldsymbol{x})K(\boldsymbol{x},\boldsymbol{x}_j)\varphi(\boldsymbol{x}_j), \quad \varphi\in C(\mathbb{S}^2), \quad \boldsymbol{x}\in\mathbb{S}^2.$$

**2.3.1. Assumptions on the integral equation.** Throughout this paper, it is assumed that the homogeneous equation corresponding to (1.1) has no nontrivial solution. Then, the Riesz theory confirms that the inhomogeneous equation (1.1) has a unique solution continuously depending on $f$.

The common theoretical setting for the convergence study is the Banach space $C(\mathbb{S}^2)$. The kernel $h$ is assumed to be weakly singular; i.e., $h$ is continuous for all $\boldsymbol{x},\boldsymbol{y}\in\mathbb{S}^2$ with $\boldsymbol{x}\neq\boldsymbol{y}$, and there exist positive constants $M$ and $\alpha\in(0,2]$ such that $|h(|\boldsymbol{x}-\boldsymbol{y}|)|\leq M|\boldsymbol{x}-\boldsymbol{y}|^{\alpha-2}$. These conditions ensure that the integral operator $A$ is a compact integral operator on $C(\mathbb{S}^2)$; see proofs in [22, section 2.4]. The detailed introduction on the analysis in this common setting can be found in, e.g., [22, Chapters 2 and 3].

However, in this paper, we need the following stronger assumptions on $h$ for our theoretical analysis:

- regarded as a function of $\boldsymbol{y}\in\mathbb{S}^2$ for all $\boldsymbol{x}\in\mathbb{S}^2$, $h(|\boldsymbol{x}-\boldsymbol{y}|)$ is assumed to satisfy

$$|h(|\boldsymbol{x}-\cdot|)|\in L^1(\mathbb{S}^2)\cap L^2(\mathbb{S}^2);\ \textit{and} \tag{2.13}$$

- regarded as a univariate function of $t\in(-1,1)$, $h(2^{1/2}(1-t)^{1/2})$ is assumed to satisfy

$$h(2^{1/2}(1-t)^{1/2})\in L^1(-1,1)\cap L^2(-1,1). \tag{2.14}$$

As we will see in the subsequent section, the $L^2$ assumption enables us to apply the $L^2$ theory of hyperinterpolation [3, 4, 5, 34].

**2.3.2. Assumptions on hyperinterpolation.** In the implementation of hyperinterpolation, we assume all quadrature weights to be positive, i.e., $w_j>0$ for all $j=1,2,\ldots,m$, and they satisfy $\sum_{j=1}^m w_j < C_w$ for some constant $C_w>0$. Specifically, if the quadrature exactness is of degree at least 1, then $C_w=|\mathbb{S}^2|$.

Additionally, we assume that the quadrature rule (1.4) used in the implementation of the numerical schemes (2.11) and (2.12) satisfies the Marcinkiewicz–Zygmund property (2.3).

**2.3.3. Assumptions on the numerical scheme (1.2).** For our analysis of the Nyström scheme (1.2), we strengthen the above assumption as follows. Given a sequence $\{I_n\}_{n\in\mathbb{N}_0}$ of quadrature rules (1.4), we assume either of the following holds:

- each $I_n$ satisfies the Marcinkiewicz–Zygmund property (2.3) of degree $n$ with an associated constant $\eta(n)\in(0,1)$, where $\eta(n)\to0$ as $n\to\infty$;
- each $I_n$ fulfills the quadrature exactness (2.2) of degree $2n$.

In the first case of our assumption, for scattered quadrature points, condition (2.5) ensures that the Marcinkiewicz–Zygmund property (2.3) holds by requiring the number of points $m$ to satisfy the asymptotic relation $m\gtrsim(n/\eta(n))^2$. If the points are instead i.i.d. drawn from the measure $d\omega$, condition (2.6) ensures (2.3) by requiring $m\geq\bar{c}n^2\log(n)/\eta(n)^2$.

**2.4. Evaluation of modified moments using the Funk–Hecke formula.** Efficient methods for evaluating the weights (2.10) are crucial for implementing the proposed numerical scheme. This in turn requires an effective approach to evaluate the modified moments $\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y})$. To this end, we apply the Funk–Hecke formula (see, e.g., [9, Theorem 2.22]).

LEMMA 2.2 (Funk–Hecke formula). *Let* $g \in L^1(-1,1)$ *and* $\boldsymbol{x} \in \mathbb{S}^2$. *Then,*

$$\int_{\mathbb{S}^2} g(\boldsymbol{x}\cdot\boldsymbol{y})Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y}) = \left(2\pi\int_{-1}^{1} g(t)P_\ell(t)dt\right)Y_{\ell,k}(\boldsymbol{x}),$$

*where* $P_\ell(t)$ *is the standard Legendre polynomial of degree* $\ell$.

Recall the definition of the geodesic distance $|\boldsymbol{x}-\boldsymbol{y}| = [2(1-\boldsymbol{x}\cdot\boldsymbol{y})]^{1/2}$. We can write $h(|\boldsymbol{x}-\boldsymbol{y}|)$ as $h(2^{1/2}(1-\boldsymbol{x}\cdot\boldsymbol{y})^{1/2})$. Since we have assumed that $h(2^{1/2}(1-t)^{1/2}) \in L^1(-1,1)$, by the Funk–Hecke formula, we have

$$\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y}) = \mu_\ell Y_{\ell,k}(\boldsymbol{x}), \tag{2.15}$$

where $\mu_\ell := 2\pi\int_{-1}^{1} h(2^{1/2}(1-t)^{1/2})P_\ell(t)dt$.

PROPOSITION 2.3 (Computational complexity). *We count floating-point operations (flops) required for implementing the two-stage scheme* (2.11) *and* (2.12). *We assume that moments* $\mu_\ell$ *and weights* $\omega_j$ *are available, and the evaluation of* $f$, $K$, *and* $Y_{\ell,k}$ *for each input costs one flop. Then,*

- *computing the weight* (2.10) *for each* $\boldsymbol{x} \in \mathbb{S}^2$ *using the Funk–Hecke formula requires approximately* $3(n+1)^2+1$ *flops in arithmetic operations and* $2(n+1)^2$ *in function evaluations;*
- *solving the linear system* (2.8) *requires approximately* $\frac{2}{3}m^3+2m^2$ *flops using Gaussian elimination or LU decomposition and* $m^2(5(n+1)^2+1)+2m^2+m$ *flops in function evaluations and additional arithmetic operations;*
- *evaluating the numerical solution* (2.12) *for each* $\boldsymbol{x} \in \mathbb{S}^2$ *requires approximately* $3m+1$ *flops in arithmetic operations and* $m(5(n+1)^2+1)+m+1$ *in function evaluations.*

*Thus, the evaluation of* $\varphi_n(\boldsymbol{x})$ *for an arbitrary* $\boldsymbol{x} \in \mathbb{S}^2$ *requires* $\mathcal{O}(m^3+m^2n^2)$ *flops.*

## 3. Error analysis.

Now, we present the error analysis for our Nyström scheme (1.2) with weights (2.10) in solving the integral equation (1.1). Recall our assumptions in section 2.3.3.

THEOREM 3.1 (Main theorem). *Consider a sequence* $\{I_n\}_{n\in\mathbb{N}_0}$ *of quadrature rules* (1.4) *where each* $I_n$ *satisfies the Marcinkiewicz–Zygmund property* (2.3) *of degree* $n$ *with an associated constant* $\eta(n) \in (0,1)$. *Assume that* $\eta(n) \to 0$ *as* $n \to \infty$. *Given the assumptions of section* 2.3.1, *there exists* $N \in \mathbb{N}_0$ *such that for all* $n \geq N$, *the Nyström scheme* (1.2), *with weights* (2.10) *computed using the quadrature rule* $I_n$, *produces an approximate solution* $\varphi_n$ *satisfying*

$$\|\varphi_n\|_{L^\infty} \leq C_1(n,h)\|f\|_{L^\infty}, \tag{3.1}$$

*where* $C_1(n,h) > 0$ *is a constant that decreases as* $n$ *grows. Moreover, let* $\boldsymbol{x}_0 \in \mathbb{S}^2$ *be the point where* $|(A_n\varphi)(x)-(A\varphi)(x)|$ *attains its maximum. Then, we have*

$$\|\varphi_n-\varphi\|_{L^\infty} \leq C_2(n,h)\left(E_n(K(\boldsymbol{x}_0,\cdot)\varphi) + \sqrt{\eta(n)^2+4\eta(n)}\|\chi^*_{\boldsymbol{x}_0}\|_{L^2}\right), \tag{3.2}$$

*where* $C_2(n,h) > 0$ *is another constant that decreases as* $n$ *grows, the best approximation is given by*

$$E_n(K(\boldsymbol{x}_0,\cdot)\varphi) := \inf_{\chi\in\mathbb{P}_n(\mathbb{S}^2)} \|K(\boldsymbol{x}_0,\cdot)\varphi-\chi\|_{L^\infty},$$

*and* $\chi^*_{\boldsymbol{x}_0} \in \mathbb{P}_n(\mathbb{S}^2)$ *denotes the best approximation polynomial of* $K(\boldsymbol{x}_0,\cdot)\varphi$ *in* $\mathbb{P}_n(\mathbb{S}^2)$.

*Remark* 3.2. The term $\|\chi^*_{\boldsymbol{x}_0}\|_{L^2}$, though it may vary with $n$, is uniform bounded in the sense that

$$\|\chi^*_{\boldsymbol{x}_0}\|_{L^2} \le 2\sqrt{4\pi} \sup_{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{S}^2\times\mathbb{S}^2} |K(\boldsymbol{x},\boldsymbol{y})|\|\varphi\|_{L^\infty} < \infty \tag{3.3}$$

for all $n\in\mathbb{N}_0$. To see this, we note that

$$\|\chi^*_{\boldsymbol{x}_0}\|_{L^2} \le \sqrt{4\pi}\|\chi^*_{\boldsymbol{x}_0}\|_{L^\infty} \le \sqrt{4\pi}\,(E_n(K(\boldsymbol{x}_0,\cdot)\varphi) + \|K(\boldsymbol{x}_0,\cdot)\varphi\|_{L^\infty}).$$

Since $0\in\mathbb{P}_n$ and the best approximation error is the infimum over all $\chi\in\mathbb{P}_n$, we have

$$E_n(K(x_0,\cdot)\varphi) = \inf_{\chi\in\mathbb{P}_n} \|K(x_0,\cdot)\varphi - \chi\|_{L^\infty} \le \|K(x_0,\cdot)\varphi\|_\infty.$$

Together with $\|K(x_0,\cdot)\varphi\|_\infty \le \sup_{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{S}^2\times\mathbb{S}^2}|K(\boldsymbol{x},\boldsymbol{y})|\|\varphi\|_{L^\infty}$, we have the desired finite result. Therefore, $\sqrt{\eta(n)^2+4\eta(n)}\|\chi^*_{\boldsymbol{x}_0}\|_{L^2}\to 0$ as $n\to\infty$.

The proof of Theorem 3.1 naturally extends to quadrature rules $I_n$ that are exact for polynomials of degree up to $2n$. Indeed, the condition $\eta(n)\to 0$ as $n\to\infty$ in the general case reduces to $\eta(n)=0$ for all $n$ in this exact quadrature setting.

COROLLARY 3.3. *Let $\{I_n\}_{n\in\mathbb{N}_0}$ be a sequence of quadrature rules* (1.4) *with each $I_n$ satisfying the exactness condition* (2.2) *for polynomials of degree up to $2n$. Then, the Nyström scheme* (1.2) *with weights* (2.10) *computed via $I_n$ satisfies all results of Theorem* 3.1*, where we take $\eta(n)=0$ for all $n$.*

**3.1. Proof of the main theorem.** The integral equation (1.1) can be written as an operator equation

$$\varphi - A\varphi = f, \tag{3.4}$$

and its solution is approximated using our Nyström scheme (1.2) with weights (2.10) by the solution of

$$\varphi_n - A_n\varphi_n = f. \tag{3.5}$$

Subtracting (3.4) from (3.5) gives

$$(I-A_n)(\varphi_n-\varphi) = (A_n - A)\varphi. \tag{3.6}$$

The existence of $\varphi_n$ and the error bound of $\|\varphi_n-\varphi\|_{L^\infty}$ depend on the existence and boundedness of $(I-A_n)^{-1}$. This fact can be confirmed by the following lemma. Note that $\|A\|$ denotes the $L^\infty$ operator norm for an operator $A: C(\mathbb{S}^2)\to C(\mathbb{S}^2)$.

LEMMA 3.4. *If the operator $A_n$ satisfies*

$$\|(I-A)^{-1}(A_n-A)A_n\| < 1, \tag{3.7}$$

*then the inverse operators $(I-A_n)^{-1}: C(\mathbb{S}^2)\to C(\mathbb{S}^2)$ exist and are bounded by*

$$\|(I-A_n)^{-1}\| \le \frac{1+\|(I-A)^{-1}A_n\|}{1-\|(I-A)^{-1}(A_n-A)A_n\|}. \tag{3.8}$$

*Moreover, for solutions of* (3.4) *and* (3.5)*, we have the error estimate*

$$\|\varphi_n-\varphi\|_{L^\infty} \le \frac{1+\|(I-A)^{-1}A_n\|}{1-\|(I-A)^{-1}(A_n-A)A_n\|}\|(A_n-A)\varphi\|_{L^\infty}. \tag{3.9}$$

*Proof.* By Riesz's theorem (cf. [22, Theorem 3.4]), (3.4) has a unique solution $\varphi$ continuously depending on $f$. That is, the inverse operator $(I-A)^{-1}: C(\mathbb{S}^2)\to C(\mathbb{S}^2)$ exists and is bounded. Then, the identity

$$(I-A)^{-1} = I + (I-A)^{-1}A$$

suggests that $B_n := I + (I-A)^{-1}A_n$ is an approximate inverse for $I-A_n$. Note that

$$\begin{aligned}(I-A)B_n(I-A_n) &= (I-A)(I+(I-A)^{-1}A_n - A_n - (I-A)^{-1}A_nA_n)\\ &= (I-A) + A_n - (I-A)A_n - A_nA_n\\ &= (I-A) - (A_n - A)A_n,\end{aligned}$$

which is equivalent to

$$B_n(I-A_n) = I - S_n, \tag{3.10}$$

where $S_n := (I-A)^{-1}(A_n-A)A_n$. From assumption (3.7) that $\|S_n\| < 1$, the Neumann series implies that $(I-S_n)^{-1}$ exists and is bounded by

$$\|(I-S_n)^{-1}\| \le \frac{1}{1-\|S_n\|}.$$

Therefore, (3.10) suggests that $I-A_n$ is an injection and then, since $A_n$ is assumed to be compact, $(I-A_n)^{-1}$ exists. Equation (3.10) also yields

$$(I-A_n)^{-1} = (I-S_n)^{-1}B_n,$$

leading to the estimate (3.8). The error estimate (3.9) then follows from (3.6). ∎

*Remark* 3.5. This lemma is motivated by the works of Brakhage [11] and Anselone and Moore [7]. In their works, (3.7) is ensured by assuming the sequence $\{A_n\}$ to be *collectively compact* and *pointwise convergent,* essentially relying on a convergent quadrature rule (1.4). The whole procedure can be also found in Kress's monograph [22] on integral equations. In our work, instead of working with a convergent quadrature rule, we try to ensure (3.7) by controlling the geometry of $\mathcal{X}_m$ and choosing an appropriate degree $n$ of hyperinterpolation. The assumption (3.7) is not ensured by the convergence of quadrature rules but the convergence of hyperinterpolation. The details are provided below.

The assumption (3.7) is ensured if we can determine $N\in\mathbb{N}_0$ such that for all $n\ge N$,

$$c\|(A_n-A)A_n\| < 1, \tag{3.11}$$

where $c$ is the upper bound of $\|(I-A)^{-1}\|$. For this goal, we first investigate some properties of $\{A_n\}_{n\in\mathbb{N}_0}$.

LEMMA 3.6. *The sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is collectively compact, i.e., for each bounded $U\subset C(\mathbb{S}^2)$, the image set $\{A_n\varphi:\varphi\in U, n\in\mathbb{N}_0\}$ is relatively compact.*

*Proof.* Recall the derivation process (2.9) of $W_j(\boldsymbol{x})$. We replace $K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})$ with a continuous function $g(\boldsymbol{y})$ with $\|g\|_{L^\infty}=1$ that satisfies, for $j=1,2,\ldots,m(n)$,

$$g(\boldsymbol{x}_j) = \begin{cases}1 & \text{if } W_j(\boldsymbol{x})\ge 0,\\ -1 & \text{if } W_j(\boldsymbol{x}) < 0.\end{cases}$$

Then, for all $\boldsymbol{x}\in\mathbb{S}^2$ and all $n\in\mathbb{N}_0$,

$$\begin{aligned}\sum_{j=1}^{m(n)}|W_j(\boldsymbol{x})| &= \sum_{j=1}^{m(n)} W_j(\boldsymbol{x})g(\boldsymbol{x}_j) = \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)\mathcal{L}_n g(\boldsymbol{y})d\omega(\boldsymbol{y})\\ &\le \left(\int_{\mathbb{S}^2}|h(|\boldsymbol{x}-\boldsymbol{y}|)|^2 d\omega(\boldsymbol{y})\right)^{1/2}\|\mathcal{L}_n g\|_{L^2}\\ &\le \left(2\pi\int_{-1}^{1}|h(2^{1/2}(1-t)^{1/2})|^2dt\right)^{1/2}\sqrt{1+\eta(n)}\left(\sum_{j=1}^{m(n)} w_j\right)^{1/2}\le C\end{aligned}$$

for some constant $C>0$, where we apply the Cauchy–Schwarz inequality, the Funk–Hecke formula, and the stability result (2.7) of hyperinterpolation. Thus, we can estimate

$$\|A_n\varphi\|_{L^\infty}\le C\max_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{S}^2}|K(\boldsymbol{x},\boldsymbol{y})|\|\varphi\|_{L^\infty}. \tag{3.12}$$

Similarly, defining another $g$ for $W_j(\tilde{\boldsymbol{x}})-W_j(\boldsymbol{x})$, we have, for all $\tilde{\boldsymbol{x}},\boldsymbol{x}\in\mathbb{S}^2$ and all $\gamma\in\Gamma$,

$$\begin{aligned}\sum_{j=1}^{m(n)}|W_j(\tilde{\boldsymbol{x}})-W_j(\boldsymbol{x})| &= \sum_{j=1}^{m(n)}(W_j(\tilde{\boldsymbol{x}})-W_j(\boldsymbol{x}))g(\boldsymbol{x}_j)\\ &=\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\left(\int_{\mathbb{S}^2}(h(|\tilde{\boldsymbol{x}}-\boldsymbol{y}|)-h(|\boldsymbol{x}-\boldsymbol{y}|))\,Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y})\right)\langle g,Y_{\ell,k}\rangle_{m(n)}\\ &=\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\mu_\ell\,(Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x}))\,\langle g,Y_{\ell,k}\rangle_{m(n)}\\ &\le\left(\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\mu_\ell^2\,(Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x}))^2\right)^{1/2}\|\mathcal{L}_n g\|_{L^2},\end{aligned} \tag{3.13}$$

where $\mu_\ell=2\pi\int_{-1}^{1}h(2^{1/2}(1-t)^{1/2})P_\ell(t)dt$.

For this upper bound, note that the latter factor $\|\mathcal{L}_n g\|_{L^2}<\infty$, regardless of the values of $n$, $\eta(n)$, and $m(n)$, as shown in the stability result (2.7) of hyperinterpolation. Then,

$$\sup_{n\in\mathbb{N}_0}\|\mathcal{L}_n g\|_{L^2}<\infty. \tag{3.14}$$

As for the former factor, we are interested in the limit of the series as $\tilde{\boldsymbol{x}}\to\boldsymbol{x}$. Since the spherical harmonics $Y_{\ell,k}$ are continuous functions on the sphere, it follows that for each fixed $\ell\ge 0$ and $1\le k\le 2\ell+1$, we have the pointwise convergence

$$\mu_\ell^2\,(Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x}))^2\to 0\quad\text{as}\quad\tilde{\boldsymbol{x}}\to\boldsymbol{x}.$$

To control the infinite sum, we first apply the addition theorem of spherical harmonics (see, e.g., [9]) that for each $\ell\in\mathbb{N}_0$,

$$\sum_{k=1}^{2\ell+1}|Y_{\ell,k}(\boldsymbol{x})|^2=\frac{2\ell+1}{4\pi},$$

and we obtain

$$\sum_{k=1}^{2\ell+1}\left|Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x})\right|^2\leq 2\sum_{k=1}^{2\ell+1}\left(\left|Y_{\ell,k}(\tilde{\boldsymbol{x}})\right|^2+\left|Y_{\ell,k}(\boldsymbol{x})\right|^2\right)\leq\frac{4(2\ell+1)}{4\pi}.$$

We then examine the coefficients $\mu_\ell=2\pi\langle h(\sqrt{2(1-\cdot)}),P_\ell\rangle$. Using properties of Legendre polynomials, namely, $\|P_\ell\|_{L^\infty}\leq 1$ and $\langle P_\ell,P_{\ell'}\rangle=2/(2\ell+1)\delta_{\ell\ell'}$ for any $\ell,\ell'\geq 0$, Parseval's identity yields the exact $L^2$ estimate

$$\sum_{\ell=0}^{\infty}(2\ell+1)\mu_\ell^2=8\pi^2\|h\|_{L^2}^2<\infty.$$

Thus, there exists a summable double sequence $\{b_{\ell,k}\}$ such that

$$\mu_\ell^2\left(Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x})\right)^2\leq b_{\ell,k}$$

and

$$\sum_{\ell=1}^{\infty}\sum_{k=1}^{2\ell+1}b_{\ell,k}\leq\frac{1}{\pi}\sum_{\ell=1}^{\infty}(2\ell+1)\mu_\ell^2<\infty.$$

Then, by the counting measure version of the Dominated Convergence Theorem, we have

$$\lim_{\tilde{\boldsymbol{x}}\to\boldsymbol{x}}\sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1}\mu_\ell^2\left(Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x})\right)^2=0. \tag{3.15}$$

Since terms are all nonnegative, we have

$$\sup_{n\in\mathbb{N}_0}\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\mu_\ell^2\left(Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x})\right)^2=\sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1}\mu_\ell^2\left(Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x})\right)^2. \tag{3.16}$$

Together with (3.13), (3.14), and (3.16), we have

$$0\leq\sup_{n\in\mathbb{N}_0}\sum_{j=1}^{m(n)}\left|W_j(\tilde{\boldsymbol{x}})-W_j(\boldsymbol{x})\right|\lesssim\left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1}\mu_\ell^2\left(Y_{\ell,k}(\tilde{\boldsymbol{x}})-Y_{\ell,k}(\boldsymbol{x})\right)^2\right)^{1/2},$$

where the last estimate is due to (3.14). By (3.15), the squeeze theorem implies

$$\lim_{\tilde{\boldsymbol{x}}\to\boldsymbol{x}}\sup_{n\in\mathbb{N}_0}\sum_{j=1}^{m(n)}\left|W_j(\tilde{\boldsymbol{x}})-W_j(\boldsymbol{x})\right|=0 \tag{3.17}$$

for all $\boldsymbol{x}\in\mathbb{S}^2$. Then, from

$$\begin{aligned}(A_n\varphi)(\tilde{\boldsymbol{x}})-(A_n\varphi)(\boldsymbol{x})=&\sum_{j=1}^{m(n)}W_j(\tilde{\boldsymbol{x}})(K(\tilde{\boldsymbol{x}},\boldsymbol{x}_j)-K(\boldsymbol{x},\boldsymbol{x}_j))\varphi(\boldsymbol{x}_j)\\&+\sum_{j=1}^{m(n)}(W_j(\tilde{\boldsymbol{x}})-W_j(\boldsymbol{x}))K(\boldsymbol{x},\boldsymbol{x}_j)\varphi(\boldsymbol{x}_j),\end{aligned}$$

we can estimate

$$
\begin{aligned}
|(A_n\varphi)(\tilde{\boldsymbol{x}}) - (A_n\varphi)(\boldsymbol{x})| \leq & C \max_{\boldsymbol{y}\in\mathbb{S}^2} |K(\tilde{\boldsymbol{x}},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})| \|\varphi\|_{L^\infty} \\
& + \max_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{S}^2} |K(\boldsymbol{x},\boldsymbol{y})| \sup_{n\in\mathbb{N}_0} \sum_{j=1}^{m(n)} |W_j(\tilde{\boldsymbol{x}}) - W_j(\boldsymbol{x})| \|\varphi\|_{L^\infty}
\end{aligned} \tag{3.18}
$$

for all $\tilde{\boldsymbol{x}},\boldsymbol{x}\in\mathbb{S}^2$. Now, let $U\subset C(\mathbb{S}^2)$ be bounded. Then, from (3.12), we see that $\{A_n\varphi : \varphi\in U, n\in\mathbb{N}_0\}$ is bounded. Let $\boldsymbol{x}\in\mathbb{S}^2$ be an arbitrary point on $\mathbb{S}^2$. For any $\varepsilon>0$, it follows from (3.17) that there exists $\delta_1>0$ such that for all $\tilde{x}$ in the $\delta_1-$neighborhood of $x$,

$$
\sup_{n\in\mathbb{N}_0} \sum_{j=1}^{m(n)} |W_j(\tilde{\boldsymbol{x}}) - W_j(\boldsymbol{x})| < \frac{\varepsilon}{2\max_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{S}^2}|K(\boldsymbol{x},\boldsymbol{y})|\|\varphi\|_{L^\infty}}.
$$

Since $K$ is continuous on the compact set $\mathbb{S}^2\times\mathbb{S}^2$, it is uniformly continuous by the Heine–Cantor theorem. Then, there exists $\delta_2>0$ such that for all $\tilde{x}$ in the $\delta_2-$neighborhood of $x$,

$$
\max_{\boldsymbol{y}\in\mathbb{S}^2} |K(\tilde{\boldsymbol{x}},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})| < \frac{\varepsilon}{2C\|\varphi\|_{L^\infty}}.
$$

Now, from estimate (3.18), for any $\varepsilon>0$, taking $\delta=\min(\delta_1,\delta_2)$, we conclude that for all $\tilde{x}$ in the $\delta-$neighborhood of $x$,

$$
|(A_n\varphi)(\tilde{\boldsymbol{x}}) - (A_n\varphi)(\boldsymbol{x})| < \varepsilon.
$$

This holds uniformly in $n\in\mathbb{N}_0$ and for all $\varphi\in U$, which implies that $\{A_n\varphi : \varphi\in U, n\in\mathbb{N}_0\}$ is pointwise equicontinuous at $\boldsymbol{x}$. Since $\mathbb{S}^2$ is compact, pointwise equicontinuity can be upgraded to uniform equicontinuity on $\mathbb{S}^2$. Then, by the Arzelà–Ascoli theorem (see, e.g., [22, Theorem 1.18]), $\{A_n\varphi : \varphi\in U, n\in\mathbb{N}_0\}$ is relatively compact, and hence the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is collectively compact. ∎

In the following lemma, we investigate the limiting case where $n\to\infty$.

LEMMA 3.7. *The sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is pointwise convergent as $n\to\infty$, i.e., $A_n\varphi\to A\varphi$ for all $\varphi\in C(\mathbb{S}^2)$ as $n\to\infty$.*

*Proof.* For fixed $\varphi\in C(\mathbb{S}^2)$, we show that the sequence $\{A_n\varphi\}_{n\in\mathbb{N}_0}$ is pointwise convergent as $n\to\infty$, i.e., $(A_n\varphi)(\boldsymbol{x})\to(A\varphi)(\boldsymbol{x})$ in the limiting case, for all $\boldsymbol{x}\in\mathbb{S}^2$. Indeed, for each $\boldsymbol{x}\in\mathbb{S}^2$, the $L^2$ error analysis of hyperinterpolation indicates that

$$
\begin{aligned}
&|(A_n\varphi)(\boldsymbol{x}) - (A\varphi)(\boldsymbol{x})| \\
&= \left| \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)\mathcal{L}_n\left(K(\boldsymbol{x},\cdot)\varphi\right)(\boldsymbol{y})d\omega(\boldsymbol{y}) - \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})d\omega(\boldsymbol{y}) \right| \\
&\leq \|h(\boldsymbol{x},\cdot)\|_{L^2} \|\mathcal{L}_n\left(K(\boldsymbol{x},\cdot)\varphi\right) - K(\boldsymbol{x},\cdot)\varphi\|_{L^2} \\
&\leq C(n,h)\left(E_n(K(\boldsymbol{x},\cdot)\varphi) + \sqrt{\eta(n)^2+4\eta(n)}\|\chi^*_{\boldsymbol{x}}\|_{L^2}\right)
\end{aligned} \tag{3.19}
$$

for some bounded constant $C(n,h)>0$, where

$$
E_n(K(\boldsymbol{x},\cdot)\varphi) := \inf_{\chi\in\mathbb{P}_n(\mathbb{S}^2)} \|K(\boldsymbol{x},\cdot)\varphi - \chi\|_{L^\infty},
$$

and $\chi_{\boldsymbol{x}}^* \in \mathbb{P}_n(\mathbb{S}^2)$ denotes the best approximation polynomial of $K(\boldsymbol{x},\cdot)\varphi$ in $\mathbb{P}_n(\mathbb{S}^2)$ in the sense of $\|K(\boldsymbol{x},\cdot)\varphi - \chi_{\boldsymbol{x}}^*\|_{L^\infty} = E_n(K(\boldsymbol{x},\cdot)\varphi)$.

To claim the pointwise convergence of the sequence $\{A_n\varphi\}_{n\in\mathbb{N}_0}$ as $n\to\infty$, we note that due to the uniform continuity of $K(x,y)$ on $\mathbb{S}^2\times\mathbb{S}^2$, the uniform boundedness and continuity of $K(\boldsymbol{x},\cdot)\varphi$ is not disrupted, and thus the Weierstrass Approximation Theorem can still apply. We have $E_n(K(\boldsymbol{x},\cdot)\varphi)\to 0$ as $n\to\infty$ for all $\boldsymbol{x}\in\mathbb{S}^2$. Besides, although $\chi_{\boldsymbol{x}}^*$ depends on $\boldsymbol{x}$, $\|\chi_{\boldsymbol{x}}^*\|_{L^2}$ is finite for all $\boldsymbol{x}\in\mathbb{S}^2$ and all $n\in\mathbb{N}_0$, as shown in (3.3). Thus, $\sqrt{\eta(n)^2+4\eta(n)\|\chi_{\boldsymbol{x}}^*\|_{L^2}}\to 0$ as $n\to\infty$; recall that $\eta(n)\to 0$ as $n\to\infty$.

As shown in the above proof of Lemma 3.6, the sequence $\{A_n\varphi\}_{n\in\mathbb{N}_0}$ is equicontinuous. Hence, by the compactness of $\mathbb{S}^2$, the convergence of this sequence can be upgraded to uniform convergence, i.e., for fixed $\varphi\in C(\mathbb{S}^2)$, $\|A_n\varphi - A\varphi\|_{L^\infty}\to 0$ as $n\to\infty$. Therefore, we have the pointwise convergence of the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ of numerical integration operators, that is, $A_n\varphi\to A\varphi$ for all $\varphi\in C(\mathbb{S}^2)$ as $n\to\infty$. $\square$

Restricting to a compact set $U\subset C(\mathbb{S}^2)$, the next lemma shows that the pointwise convergence of $\{A_n\}_{n\in\mathbb{N}_0}$ can be further upgraded to uniform convergence.

LEMMA 3.8. *Let $U$ be a compact subset of $C(\mathbb{S}^2)$. Then, $\{A_n\}_{n\in\mathbb{N}_0}$ has uniform convergence on $U$ to $A$, i.e.,* $\sup_{\varphi\in U}\|A_n\varphi - A\varphi\|_{L^\infty}\to 0$ *as $n\to\infty$.*

*Proof.* From the bound (3.12) on each $A_n$, the uniform boundedness principle results in a uniform bound on the operator norms of all $A_n$, i.e., there exists some constant $C>0$ such that $\|A_n\|\le C$ for all $n\in\mathbb{N}_0$. For any $\varepsilon>0$, consider open balls $B(\varphi,r)=\{\psi\in C(\mathbb{S}^2):\|\varphi-\psi\|_{L^\infty}<r\}$ with center $\varphi\in C(\mathbb{S}^2)$ and radius $r=\varepsilon/(3C)$. Due to the compactness of $U$, there exists a finite covering such that $U\subset\bigcup_{k=1}^M B(\varphi_k,r)$. Pointwise convergence of $\{A_n\}$, as shown in Lemma 3.7, suggests that there exists $N(\varepsilon)\in\mathbb{N}_0$ such that $\|A_n\varphi_k - A\varphi_k\|_{L^\infty}<\varepsilon/3$ for all $n\ge N(\varepsilon)$ and for all $k=1,\dots,M$. Now, let $\varphi\in U$ be arbitrary. Then, $\varphi$ is contained in some ball $B(\varphi_k,r)$ with center $\varphi_k$. Hence, for all $n\ge N(\varepsilon)$, we have

$$\begin{aligned}\|A_n\varphi - A\varphi\|_{L^\infty} &\le \|A_n\varphi - A_n\varphi_k\|_{L^\infty} + \|A_n\varphi_k - A\varphi_k\|_{L^\infty} + \|A\varphi_k - A\varphi\|_{L^\infty}\\ &\le \|A_n\|\|\varphi-\varphi_k\|_{L^\infty} + \varepsilon/3 + \|A\|\|\varphi-\varphi_k\|_{L^\infty}\\ &\le 2Cr + \varepsilon/3 = \varepsilon.\end{aligned}$$

Therefore, the convergence of $\{A_n\}_{n\in\mathbb{N}_0}$ is uniform on $U$. $\square$

LEMMA 3.9. $\|(A_n - A)A_n\|\to 0$ *as $n\to\infty$.*

*Proof.* Consider the set $U:=\{A_n\varphi:\|\varphi\|_{L^\infty}\le 1, n\in\mathbb{N}_0\}$. By Lemma 3.6, the set $U$ is relatively compact. By Lemma 3.8, the convergence $A_n\psi\to A\psi$ as $n\to\infty$ is uniform for all $\psi\in U$. Hence, for any $\varepsilon>0$, there exists $N(\varepsilon)\in\mathbb{N}_0$ such that the sequence of functions satisfies $\|(A_n-A)A_n\varphi\|_{L^\infty}<\varepsilon$ for all $n\ge N(\varepsilon)$ and all $\varphi\in C(\mathbb{S}^2)$ with $\|\varphi\|_{L^\infty}\le 1$. Therefore, the operator norms satisfy $\|(A_n-A)A_n\|\le\varepsilon$ for all $n\ge N(\varepsilon)$. $\square$

We are now prepared well for the proof of our main theorem.

*Proof of Theorem* 3.1. This theorem follows from Lemma 3.4, which guarantees the existence and boundedness of $(I-A_n)^{-1}$. The assumption (3.7) in Lemma 3.4 is ensured by (3.11), which is a direct consequence of Lemma 3.9. Therefore, we can conclude that $(I-A_n)^{-1}$ exists and is bounded, which implies the bound (3.1) of $\varphi_n$. For the error estimate (3.2), let $\boldsymbol{x}_0\in\mathbb{S}^2$ be the point where $|(A_n\varphi)(\boldsymbol{x})-(A\varphi)(\boldsymbol{x})|$ attains its maximum, that is, $|(A_n\varphi)(\boldsymbol{x}_0)-(A\varphi)(\boldsymbol{x}_0)| = \|A_n\varphi - A\varphi\|_{L^\infty}$. Note that $\boldsymbol{x}_0$ depends on $n$. Then, from the estimate (3.19), we have

$$|(A_\gamma\varphi)(\boldsymbol{x}_0)-(A\varphi)(\boldsymbol{x}_0)|\le C(n,h)\left(E_n(K(\boldsymbol{x}_0,\cdot)\varphi)+\sqrt{\eta^2+4\eta}\|\chi^*_{\boldsymbol{x}_0}\|_{L^2}\right)$$

for some bounded constant $C(m,n,\eta,h)>0$. Together with (3.9), we obtain the error estimate (3.2). ◻

**4. Implementation and examples of kernels.** The explicit construction of $W_j(\boldsymbol{x})$ relies on the explicit evaluation of the modified moments (2.15). These moments are explicitly available for some specific kernels. In this section, we list some singular kernels and elaborate the evaluation of the corresponding moments.

**4.1. Nonsingular case.** If $h\equiv 1$, since

$$\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\left(\int_{\mathbb{S}^2}h(|\boldsymbol{x}-\boldsymbol{y}|)Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y})\right)Y_{\ell,k}(\boldsymbol{x}_j)=1,$$

we have

$$W_j(\boldsymbol{x})=w_j\left(\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\left(\int_{\mathbb{S}^2}h(|\boldsymbol{x}-\boldsymbol{y}|)Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y})\right)Y_{\ell,k}(\boldsymbol{x}_j)\right)\equiv w_j.$$

**4.2. Singular cases.** We explore three common singular kernels in the context of integral equations.

**Case 1: $\boldsymbol{h(|x-y|)=|x-y|^\nu}$** with $-1<\nu<0$. The assumption $\nu>-1$ is sufficient for the $L^2$ assumption of $h$. For the case of $\nu\ge 0$, it corresponds to a continuous kernel. For the modified moments (2.15) with this specific kernel, we use the Rodrigues representation formula

$$P_\ell(t)=\frac{(-1)^\ell}{2^\ell\ell!}\left(\frac{d}{dt}\right)^\ell\left(1-t^2\right)^\ell$$

for Legendre polynomial $P_\ell(t)$ of degree $\ell$. Then, we can express the constant $\mu_\ell$ in the form of

$$\mu_\ell=2\pi\frac{(-1)^\ell}{2^\ell\ell!}\int_{-1}^{1}2^{\nu/2}(1-t)^{\nu/2}\left(\frac{d}{dt}\right)^\ell\left(1-t^2\right)^\ell dt,$$

which can be further simplified through computing the integral

$$I(\nu)=\int_{-1}^{1}(1-t)^{\nu/2}\left(\frac{d}{dt}\right)^\ell\left(1-t^2\right)^\ell dt.$$

Given the condition $\nu>-1$, we can perform integration by parts repeatedly on $I(\nu)$, and all terms containing $(1-t^2)$ vanish at $t=\pm1$. After integrating by parts $\ell$ times, we have

$$I(\nu)=\frac{\nu}{2}\left(\frac{\nu}{2}-1\right)\cdots\left(\frac{\nu}{2}-(\ell-1)\right)J(\nu)=(-1)^\ell\left(-\frac{\nu}{2}\right)_\ell J(\nu),$$

where $(x)_n:=\frac{\Gamma(x+n)}{\Gamma(x)}$ with $x>0$ stands for Pochhammer's symbol and

$$J(\nu):=\int_{-1}^{1}(1-t)^{\nu/2-\ell}\left(1-t^2\right)^\ell dt=\int_{-1}^{1}(1-t)^{\nu/2}(1+t)^\ell dt.$$

The term $J(\nu)$ can be further evaluated via the change of variables $t=2s-1$ as

$$J(\nu)=2^{\ell+\frac{\nu}{2}+1}\int_0^1(1-s)^{\frac{\nu+2}{2}-1}s^{\ell}ds=2^{\ell+\frac{\nu}{2}+1}\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\ell+1\right)}{\Gamma\left(\ell+\frac{\nu}{2}+2\right)}.$$

Therefore, the modified moment (2.15) of $h(|\boldsymbol{x}-\boldsymbol{y}|)=|\boldsymbol{x}-\boldsymbol{y}|^{\nu}$ with $-1<\nu<0$ can be evaluated analytically as

$$\mu_{\ell}=2^{\nu+2}\pi\left(-\frac{\nu}{2}\right)_{\ell}\frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\ell+\frac{\nu}{2}+2\right)}.$$

More details on this derivation can be found in [9, section 3.7.1].

**Case 2: $h(|\boldsymbol{x}-\boldsymbol{y}|)=\log|\boldsymbol{x}-\boldsymbol{y}|$.** For the modified moments (2.15) with this specific kernel, note that $\log(2^{1/2}(1-t)^{1/2})$ satisfies the requirement (2.14). Thus, we have

$$\mu_{\ell}=2\pi\int_{-1}^{1}\log(2^{1/2}(1-t)^{1/2})P_{\ell}(t)dt=\pi\int_{-1}^{1}\log(2(1-t))P_{\ell}(t)dt.$$

For $\ell=0$, we have

$$\mu_0=\pi\int_{-1}^{1}\log(2(1-t))dt=\pi\left(2\log 2+\int_{-1}^{1}\log(1-t)dt\right)=\pi(4\log 2-2).$$

For $\ell\geq 1$, we utilize the orthogonality of Legendre polynomials and the Maclaurin series of $\log(1-t)$ for

$$\mu_{\ell}=-\pi\int_{-1}^{1}\sum_{k=1}^{\infty}\frac{t^k}{k}P_{\ell}(t)dt=-\pi\sum_{k=1}^{\ell}\frac{1}{k}\int_{-1}^{1}t^kP_{\ell}(t)dt,\quad \ell=1,2,\ldots$$

From our numerical experience, replacing $\log(1-t)$ with its Maclaurin series should be necessary. To compute integrals $\int_{-1}^{1}t^kP_{\ell}(t)dt$, we work with Chebfun [19]. Here is a Matlab code segment that evaluates $\mu_{\ell}$ for $\ell=0,1,\ldots,n$.

```
x = chebfun('x');
mu(1) = pi*(4*log(2)-2);
for ell = 1:n
  mu_temp = 0;
  for k = 1:ell
    mu_temp = mu_temp + sum(x.^k.*legpoly(ell))/k;
  end
  mu(ell+1) = -pi * mu_temp;
end
```

**Case 3: $h(|\boldsymbol{x}-\boldsymbol{y}|)=|\boldsymbol{x}-\boldsymbol{y}|^{\nu_1}|\boldsymbol{x}+\boldsymbol{y}|^{\nu_2}$** with $-1\leq\nu_1,\nu_2<0$. Similar to Case 2, we compute the modified moments (2.15) with this specific kernel in terms of

$$\mu_{\ell}=2^{(\nu_1+\nu_2)/2}(2\pi)\int_{-1}^{1}(1-t)^{\nu_1/2}(1+t)^{\nu_2/2}P_{\ell}(t)dt.$$

Specifying $\nu_1$ and $\nu_2$ as `nu1` and `nu2`, a Matlab code segment for $\mu_{\ell}$ with $\ell=0,1,\ldots,n$ is provided below.

```
x = chebfun('x');
const = 2^((nu1+nu2)/2)*2*pi;
for ell = 0:n
  mu(ell+1) = const*sum((1-x).^(nu1/2).*(1+x).^(nu2/2)...
    .*legpoly(ell));
end
```

**5. Numerical experiments.** Here, we give numerical results for our two-stage scheme (2.11) and (2.12) in solving the integral equation (1.1) defined using both nonsingular kernels and the three types of singular kernels mentioned above. We investigate the following four kinds of point distributions on sphere:

- Spherical $t$-designs [17]: a set of points $\mathcal{X}_m$ with the characterizing property that an equal-weight quadrature rule in these points exactly integrates all polynomials of degree at most $t$, that is,

$$\frac{4\pi}{m}\sum_{j=1}^{m}\chi(\boldsymbol{x}_j)=\int_{\mathbb{S}^2}\chi(\boldsymbol{y})\mathrm{d}\omega(\boldsymbol{y}) \quad \forall \chi\in\mathbb{P}_t. \tag{5.1}$$

  In our experiment, we employ well conditioned spherical $t$-designs [2] to realize the approximation.
- Minimal energy points [37]: a set of points $\mathcal{X}_m$ that minimizes the Coulomb energy $\sum_{i,j=1}^{m}1/\|\boldsymbol{x}_i-\boldsymbol{x}_j\|_2$.
- Fekete points [35]: a set of points obtained by maximizing the determinant of the interpolation matrix.
- Equal area points [31]: a set of recursive zonal equal area partitions of $\mathbb{S}^2$, generated using an algorithm given in [24].

These four kinds of points are utilized in the first stage (2.11) of our scheme. For the second stage (2.12), evaluating values of the numerical solution, we adopt 5,000 uniformly distributed points on $\mathbb{S}^2$.

**Experiment 1: $\boldsymbol{h(x,y)\equiv 1}$.** We begin by testing our scheme using the case where the solution is constant, specifically $\varphi\equiv 1$. Though $h$ is nonsingular, we let the continuous kernel be defined as $K(\boldsymbol{x},\boldsymbol{y})=\sin(10|\boldsymbol{x}-\boldsymbol{y}|)$, exhibiting oscillatory behavior. Analytically, based on the Funk–Hecke formula, we determine that the inhomogeneous term $f$ should also be constant:

$$f(\boldsymbol{x},\boldsymbol{y})=1-2\pi\int_{-1}^{1}\sin\left(10\sqrt{2(1-t)}\right)dt\approx \texttt{1.455449001125579}.$$

This integral, which involves a continuous integrand, is computed using MATLAB's built-in `integral` command. We set the degree of hyperinterpolation to $n=20$ and choose the number of quadrature points as $m=(2n+1)^2$. Figure 1 illustrates the numerical solutions of the integral equation (1.1). We consider four types of quadrature points, comparing their errors with respect to the exact solution $\varphi\equiv 1$. Notably, the numerical solutions closely approximate the constant value of 1, as indicated by the color bar ranges, with the scheme utilizing spherical $t$-designs performing the best.

To further explore their comparison and validate our error analysis, we consider uniform errors ($L^\infty$ errors) of numerical solutions for varying $n$ and $m$. The first plot in Figure 2 investigates the errors of numerical solutions of the equation considered in Figure 1 with fixed $n=20$ and varying $m$ in the sense that $m=(t+1)^2$ for $t$ ranging from 15 to 35. The second plot demonstrates errors with $n$ ranging from 15 to 35 and $m$ varying corresponding to $n$ in terms of $m=(\lfloor 1.2n\rfloor+1)^2$. Figure 2 not only enriches the comparison results among quadrature points but also validates our theoretical analysis that the $L^\infty$ errors decrease as $n$ grows and the constant $\eta$ in the Marcinkiewicz–Zygmund property (2.3) decreases. The decrease in $\eta$ is a consequence of an increasing number $m$ of points for a well-distributed point set.

**Experiment 2: $\boldsymbol{h(x,y)=|x-y|^\nu}$ with $-1<\nu<0$.** We continue by considering a singular kernel $h(\boldsymbol{x},\boldsymbol{y})=|\boldsymbol{x}-\boldsymbol{y}|^\nu$ with $\nu=-0.5$. Similar to Experiment 1, we

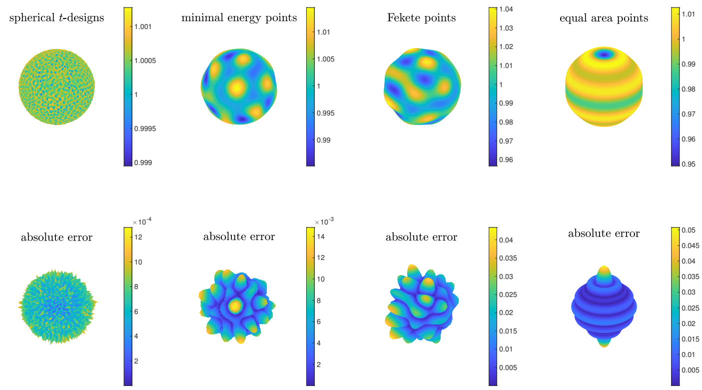


FIG. 1. *Nonsingular $h=1$ and oscillatory $K(\boldsymbol{x},\boldsymbol{y})=\sin(10|\boldsymbol{x}-\boldsymbol{y}|)$: Numerical solutions ($n=20$ and $m=(2n+1)^2$) of* (1.1) *with exact solution being constant* 1 *and their absolute errors.*

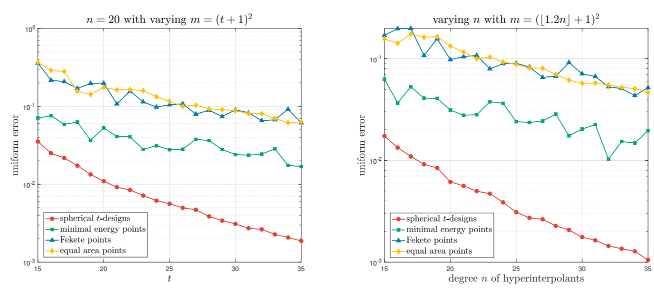


FIG. 2. *Nonsingular $h=1$ and oscillatory $K(\boldsymbol{x},\boldsymbol{y})=\sin(10|\boldsymbol{x}-\boldsymbol{y}|)$: Uniform errors with different $n$ and $m$.*

test our scheme using the case where the solution is constant, $\varphi\equiv 1$. We consider an oscillatory kernel again, $K(\boldsymbol{x},\boldsymbol{y})=\cos(10|\boldsymbol{x}-\boldsymbol{y}|)$. The inhomogeneous term

$$f(\boldsymbol{x},\boldsymbol{y})\equiv 1-2\pi\int_{-1}^{1}\left(\sqrt{2(1-t)}\right)^{-0.5}\cos\left(10\sqrt{2(1-t)}\right)dt$$

involves a singular integral. To evaluate it, we apply the substitution $u=\sqrt{2(1-t)}$, transforming the integral into

$$\int_{-1}^{1}\left(\sqrt{2(1-t)}\right)^{-0.5}\cos\left(10\sqrt{2(1-t)}\right)dt=\int_{0}^{2}u^{1/2}\cos(10u)du.$$

The resulting integrand is now continuous, allowing us to compute it accurately using `integral`. This yields the approximation $f(\boldsymbol{x},\boldsymbol{y})\approx$ `0.303738732800340`. We set the degree of hyperinterpolation to $n=20$ and choose the number of quadrature points as $m=(2n+1)^2$. Figure 3 illustrates the numerical solutions of the integral equation (1.1) with their absolute errors. These solutions are still close to the exact solution $\varphi\equiv 1$, and again, the scheme utilizing spherical $t$-designs performs the best.

Similar to Figure 2, we validate our theoretical analysis for varying $n$ and $m$ in the case of $h(\boldsymbol{x},\boldsymbol{y})=|\boldsymbol{x}-\boldsymbol{y}|^{-0.5}$. This validation is demonstrated in Figure 4.

**Experiment 3: $\boldsymbol{h(x,y)=\log|x-y|}$.** We then consider another singular kernel, the log kernel (with natural base $e$). Similar to previous experiments, we test our scheme using the case where the solution is $\varphi\equiv 1$. In this experiment, we let $K=1$.

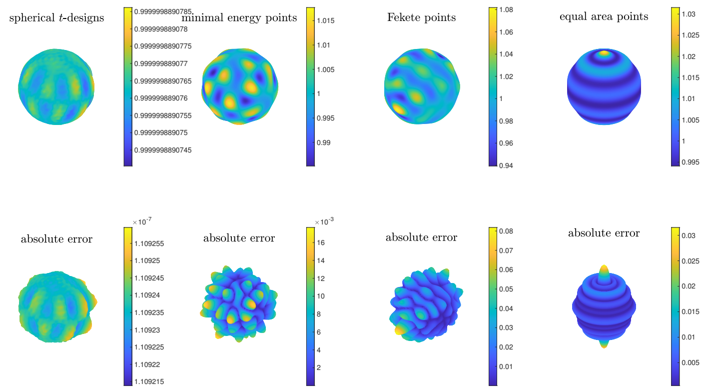


FIG. 3. *Singular* $h(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|^{-0.5}$ *and oscillatory* $K(\boldsymbol{x},\boldsymbol{y}) = \cos(10|\boldsymbol{x}-\boldsymbol{y}|)$*: Numerical solutions* ($n = 20$ *and* $m = (2n+1)^2$) *of the equation* (1.1) *with exact solution being constant* 1 *and their absolute errors.*

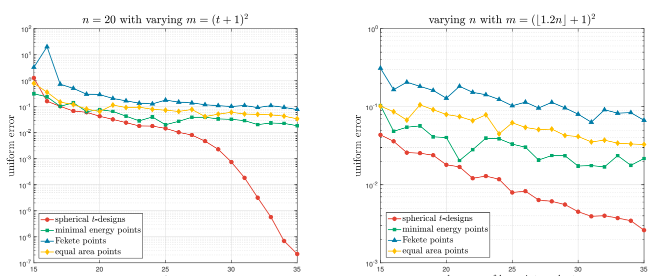


FIG. 4. *Singular* $h(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|^{-0.5}$ *and oscillatory* $K(\boldsymbol{x},\boldsymbol{y}) = \cos(10|\boldsymbol{x}-\boldsymbol{y}|)$*: Uniform errors with different* $n$ *and* $m$*.*

The reason for considering the nonoscillatory kernel $K$ is to test the accuracy of our scheme. Recall our error bound (3.2): if $K = 1$ and $\varphi = 1$, then $E_n(K(\boldsymbol{x}_0,\cdot)\varphi) = 0$ for $n \geq 1$; and if the quadrature rule is exact for polynomials of degree $2n$, then $\sqrt{\eta^2 + 4\eta}\|\chi^*\|_{L^2} = 0$. Thus, in the above situation, we should observe the error to be near machine epsilon. If $K$ contains oscillatory behavior, then similar results to the previous two experiments have been observed. In this case, note that the inhomogeneous term

$$f(\boldsymbol{x},\boldsymbol{y}) \equiv 1 - 2\pi \int_{-1}^{1} \log\left(\sqrt{2(1-t)}\right) dt = 1 - \pi(4\log 2 - 2).$$

We set the degree of hyperinterpolation to $n = 5$ and choose the number of quadrature points as $m = (2n+1)^2$. Figure 5 illustrates the numerical solutions of the integral equation (1.1) with their absolute errors. The solution using spherical $t$-design is computed with error near machine epsilon. We further explore this interesting fact with varying $n$ and $m$.

The uniform errors of computed numerical solutions with varying $n$ and $m$ for the log kernel are depicted in Figure 6. Our error analysis is validated in Figure 6 again. Notably, the error plot related to spherical $t$-designs rapidly approaches the machine epsilon, as explained before. Subsequently, this error curve starts to grow slowly due to the accumulation of round-off errors as the size of the linear system in our scheme increases.

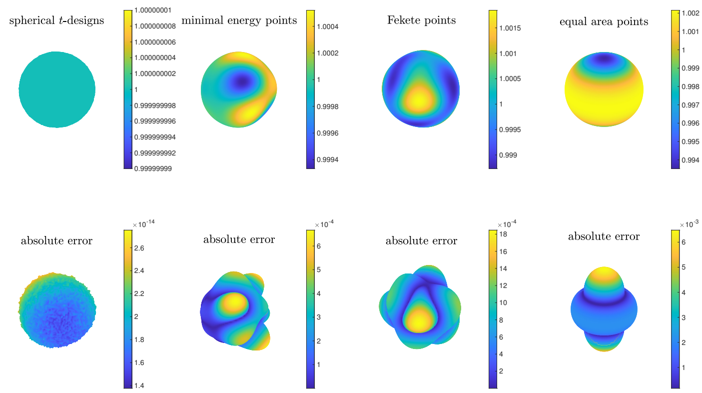


FIG. 5. *Singular* $h(\boldsymbol{x},\boldsymbol{y}) = \log|\boldsymbol{x}-\boldsymbol{y}|$ *and nonoscillatory* $K=1$*: Numerical solutions* ($n=5$ *and* $m=(2n+1)^2$) *of* (1.1) *with exact solution being constant* 1 *and their absolute errors.*

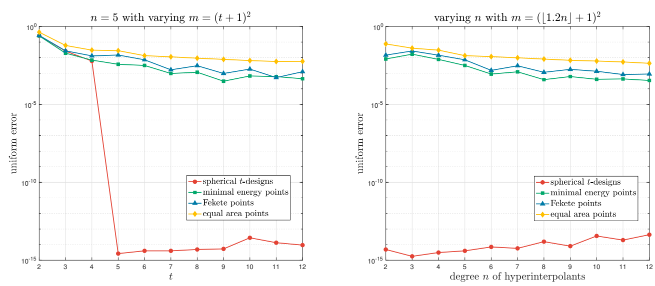


FIG. 6. *Singular* $h(\boldsymbol{x},\boldsymbol{y}) = \log|\boldsymbol{x}-\boldsymbol{y}|$ *and nonoscillatory* $K=1$*: Uniform errors with different* $n$ *and* $m$*.*

**Experiment 4:** $\boldsymbol{h(x,y) = |x-y|^{\nu_1}|x+y|^{\nu_2}}$**.** We then conduct an experiment in the same conditions as Experiment 1 but with the singular kernel modified to $|\boldsymbol{x}-\boldsymbol{y}|^{-0.5}|\boldsymbol{x}+\boldsymbol{y}|^{-0.5}$. For the inhomogeneous term in this case, we introduce a new variable $\theta$ such that $t=\cos(\theta)$, thus

$$\int_{-1}^{1}\left(\sqrt{2(1-t)}\right)^{-0.5}\left(\sqrt{2(1+t)}\right)^{-0.5}\sin\left(10\sqrt{2(1-t)}\right)dt$$
$$=\int_0^{\pi}\sin(20\sin(\theta/2))\cdot\sin^{1/2}(\theta/2)\cos^{1/2}(\theta/2)d\theta,$$

yielding $f(\boldsymbol{x},\boldsymbol{y})\approx$ `0.930837885429478`. For $n=20$, the numerical solutions and their errors are displayed in Figure 7. Errors for varying $n$ and $m$ are shown in Figure 8. The conclusions drawn are consistent with those of the previous experiments.

**6. Discussion.** In this paper, we propose a two-stage numerical scheme (2.11) and (2.12) for solving the integral equation (1.1) of the second kind on the sphere $\mathbb{S}^2$ and provide numerical analysis. This scheme combines hyperinterpolation and product integration rules to provide an explicit evaluation of the weights (2.10) $W_j(\boldsymbol{x})$. We also present numerical experiments to show the accuracy of our scheme and validate our analysis.

In the construction of $W_j(\boldsymbol{x})$, the distribution of quadrature points $\mathcal{X}_m$ and the evaluation of modified moments (2.15) are essential. Traditionally, in numerical analysis, quadrature points are chosen to ensure that the corresponding quadrature rule

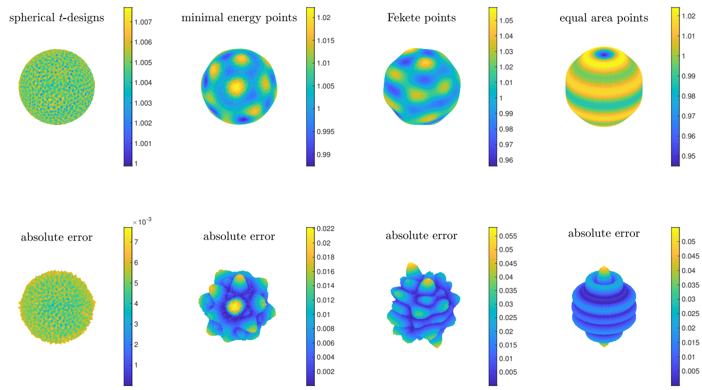


FIG. 7. *Singular* $h(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|^{-0.5}|\boldsymbol{x}+\boldsymbol{y}|^{-0.5}$ *and oscillatory* $K(\boldsymbol{x},\boldsymbol{y}) = \sin(10|\boldsymbol{x}-\boldsymbol{y}|)$*: Numerical solutions (*$n = 20$ *and* $m = (2n+1)^2$*) of* (1.1) *with exact solution being constant* 1 *and their absolute errors.*

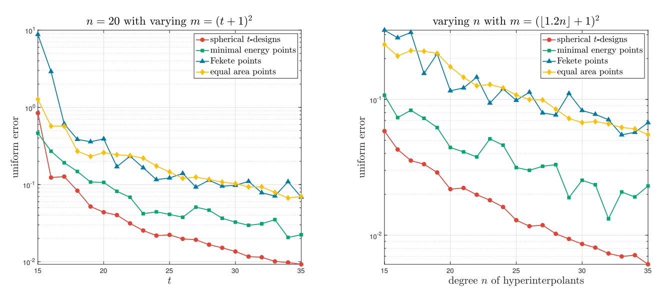


FIG. 8. *Singular* $h(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|^{-0.5}|\boldsymbol{x}+\boldsymbol{y}|^{-0.5}$ *and oscillatory* $K(\boldsymbol{x},\boldsymbol{y}) = \sin(10|\boldsymbol{x}-\boldsymbol{y}|)$*: Uniform errors with different* $n$ *and* $m$*.*

is exact for polynomials of certain degrees. While this approach allows for straightforward error bounds with respect to polynomial degrees, it offers limited choices of quadrature points. In our recent works [3, 4, 5], we have relaxed the exactness assumption to the Marcinkiewicz–Zygmund (MZ) property (2.3). This relaxation enables a broader range of quadrature points to be utilized while still deriving reasonable error bounds. Spherical $t$-designs, which provide rotationally invariant quadrature rules with exactness, perform exceptionally well. However, other quadrature points that do not guarantee exactness can also yield satisfactory results. Particularly, minimal energy points are generally unsuitable for polynomial interpolation since the corresponding Lebesgue constant is very large, as noted in An's PhD thesis [1, section 3]. Nonetheless, in this paper, we demonstrate that minimal energy points can effectively solve integral equations using the product integration method. Our method and analysis can be extended to other point sets that satisfy the MZ property (2.3), including randomly distributed points [23] or those with MZ-like properties, such as quasi-Monte Carlo (QMC) designs [12]. For a comparison between point sets satisfying the MZ property and QMC designs, we refer to our previous paper [4].

The evaluation of modified moments (2.15) in this paper is based on the Funk–Hecke formula. This formula is effective for certain special singular kernels, such as the algebraic singular kernel $|\boldsymbol{x}-\boldsymbol{y}|^{\nu}$ with $\nu < 0$, the log kernel $\log|\boldsymbol{x}-\boldsymbol{y}|$, and the extended algebraic singular kernel $|\boldsymbol{x}-\boldsymbol{y}|^{\nu_1}|\boldsymbol{x}-\boldsymbol{y}|^{\nu_2}$ with $\nu_1, \nu_2 < 0$. For cases involving both singular and oscillatory kernels, we treat the continuous oscillatory kernels as $K(\boldsymbol{x}, \boldsymbol{y})$ and approximate the product of $K$ and the solution $\varphi$ using hyperinterpolation. The degree of hyperinterpolation must be sufficiently high to capture the oscillatory behavior of $K$. The evaluation of modified moments (2.15) is a crucial topic in numerical methods for singular integral equations. Deriving analytical expressions and stable evaluation methods, particularly for mixed singular and oscillatory kernels, will significantly enhance the applicability of our method.

The theory of hyperinterpolation using quadrature points that satisfy the MZ property (2.3) plays an important role in our analysis. This theory imposes the $L^2$ assumption on the singular kernel $h$, which is the only additional assumption we have made compared to the common setting of singular integral equations. If the kernel $h$ is too singular to meet this assumption, we expect to implement transformations, such as Atkinson's transformation [8] with the choices of "grading parameters," to obtain a new integrand that is much smoother and thus fulfills the $L^2$ assumption.

Apart from quadrature methods, there are other numerical methods that have been extensively investigated in the literature of integral equations; see, e.g., [13, 22] and references therein. However, these methods have not been explored using quadrature rules that satisfy the MZ property. We hope that future contributions will address this gap.

**Acknowledgments.** We are grateful to Professor Ian H. Sloan for his encouragement and comments. We also extend our gratitude to Professor Xiaoming Yuan and the organizers and participants of the International Conference on Scientific Machine Learning (Dec 2024) at the University of Hong Kong, where a version of this work was presented. Our sincere thanks also go to the editor and anonymous referees for their careful reading of our manuscript and their many insightful comments.

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