Bypassing the quadrature exactness assumption of hyperinterpolation†

†based on joint works with Congpei An (Guizhou Univ)

Hao-Ning Wu
University of Georgia
May 21, 2025

Constructive Functions 2025
Nashville, TN

Polynomial approximation

☞ For $f \in C(\Omega)$, find an approximant $p = \sum_{\ell=1}^{d_n} c_\ell p_\ell \in \mathbb{P}_n$:

- $\Omega \subset \mathbb{R}^d$: bounded, closed subset of $\mathbb{R}^d$ or compact manifold with finite measure w.r.t a given (positive) measure $\mathrm{d}\omega$, i.e., $\int_\Omega \mathrm{d}\omega = V$.
- $\mathbb{P}_n$: space of polynomials of degree $\leq n$ over Ω
- $\{p_1, p_2, \ldots, p_{d_n}\}$: orthonormal basis of $\mathbb{P}_n$ with dim. $d_n := \dim \mathbb{P}_n$

☞ Famous Methods:

- **Polynomial interpolation**: given points $\{x_j\}_{j=1}^{d_n}$, find p such that

$$\boxed{f(x_j) = p(x_j)} = \sum_{\ell=1}^{d_n} c_\ell p_\ell(x_j), \quad j = 1, \ldots, d_n$$

  - complicated and even problematic in multivariate cases

- **Orthogonal projection**: defined as

$$\boxed{\mathcal{P}_n f := \sum_{\ell=1}^{d_n} \langle f, p_\ell \rangle p_\ell,}$$

  where $\langle f, g \rangle = \int_\Omega fg \mathrm{d}\omega$

  - non-implementable on computers

Hyperinterpolation

✍ Ian H. Sloan (in the early 1990s): Does the interpolation of functions on $\mathbb{S}^1$ have properties as good as orthogonal projection?

- on $\mathbb{S}^1$: Yes.
- on $\mathbb{S}^d$ ($d \geq 2$) and most high-dim regions: remaining **Problematic** to this day!
- Using more points than interpolation? $\rightarrow$ **hyper**interpolation

The **hyperinterpolation** of $f \in C(\Omega)$ onto $\mathbb{P}_n$ is defined as

$$\mathcal{L}_n f := \sum_{\ell=1}^{d_n} \langle f, p_\ell \rangle_m p_\ell,$$

where $\langle f, g \rangle_m := \sum_{j=1}^{m} w_j f(x_j) g(x_j)$ with all $w_j > 0$.

- $\mathcal{L}_n f$ is a discretized version of the **orthogonal projection** $\mathcal{P}_n f$.
- $\mathcal{L}_n f$ reduces to **interpolation** if the quadrature rule is d_n-point with exactness degree exceeding $2n$.

The quadrature rule $\sum_{j=1}^{m} w_j g(x_j) \approx \int_\Omega g \mathrm{d}\omega$ is said to have **exactness degree** $2n$ if

$$\sum_{j=1}^{m} w_j g(x_j) = \int_\Omega g \mathrm{d}\omega \quad \forall g \in \mathbb{P}_{2n}.$$

Caveat: Such quadrature rules (d_n-point with exactness degree at least $2n$) only exist on a few low-dimensional Ω, such as $[-1,1]$, $[-1,1]^2$, and $\mathbb{S}^1$; and they are generally not available on $[-1,1]^d$ ($d \geq 3$) or $\mathbb{S}^d$ ($d \geq 2$).

In higher dimensions, more quadrature points (than d_n) are necessary for exactness degree $2n$

Theorem (Sloan 1995)

Assume the quadrature rule has exactness degree $2n$. Then for any $f \in C(\Omega)$, its hyperinterpolant $\mathcal{L}_n f$ satisfies:

- $\mathcal{L}_n \chi = \chi$ for any $\chi \in \mathbb{P}_n$;
- $\|\mathcal{L}_n f\|_2 \leq V^{1/2} \|f\|_\infty$;
- $\|\mathcal{L}_n f - f\|_2 \leq 2V^{1/2} E_n(f)$.

Here $V = |\Omega|$ and $E_n(f) := \inf_{\chi \in \mathbb{P}_n} \|f - \chi\|_\infty$.

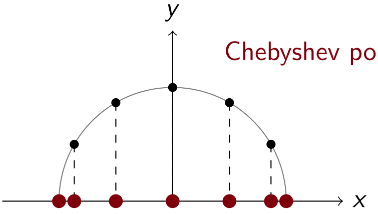
☞ The interpolation of functions on $\mathbb{S}^1$ has properties as good as orthogonal projection ✓

☞ That on $\mathbb{S}^2$ or higher dimensional spheres ?

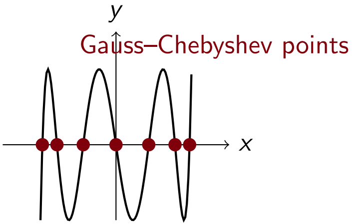
Caveat: The theorem relies on quad. exactness of degree at least $2n$:

$$\sum_{j=1}^{m} w_j g(x_j) = \int_\Omega g \mathrm{d}\omega \quad \forall g \in \mathbb{P}_{2n}.$$

On the quadrature exactness



Clenshaw–Curtis quad (1960)
$n+1$ points $\to$ n exactness degree



Gauss–Chebyshev quad (19th century)
$n+1$ points $\to$ $2n+1$ exactness degree

✍ Trefethen (2008): entered the complex plane and demonstrated for most functions (particularly those that are analytic), the Clenshaw–Curtis and Gauss quadrature rules have comparable accuracy

✍ Trefethen (2022): numerical integral is an analysis topic, while quadrature exactness is an algebraic matter

Our solution: Marcinkiewicz–Zygmund

✍ Marcinkiewicz and Zygmund (1937): There exists $\eta \in [0,1)$ such that

$$(1-\eta)\int_\Omega \chi^2 \mathrm{d}\omega_d \le \sum_{j=1}^m w_j \chi(x_j)^2 \le (1+\eta)\int_\Omega \chi^2 \mathrm{d}\omega_d \quad \forall \chi \in \mathbb{P}_n.$$

- MZ on spheres: Mhaskar, Narcowich, & Ward (2001)
- MZ on compact manifolds: Filbir & Mhaskar (2011)
- MZ on multivariate domains other than compact manifolds (balls, polytopes, cones, spherical sectors, etc.): De Marchi & Kroó (2018)

In particular, $[h_{\mathcal{X}_m} := \max_{x \in \mathbb{S}^{d-1}} \min_{x_j \in \mathcal{X}_m} \mathrm{dist}(x, x_j)]$

- MZ on compact manifolds holds if $n \lesssim \eta / h_{\mathcal{X}_m}$, where $h_{\mathcal{X}_m}$ is the mesh norm of $\{x_j\}_{j=1}^m \Rightarrow$ scattered data
- Le Gia and Mhaskar (2009): If $\{x_j\}$ are i.i.d drawn from the distribution ω_d, then there exists a constant $\bar{c} := \bar{c}(\gamma)$ such that MZ holds on $\mathbb{S}^d$ with probability $\ge 1 - \bar{c}n^{-\gamma}$ on the condition $m \ge \bar{c}n^d \log n/\eta^2 \Rightarrow$ random data and learning theory

Marcinkiewicz–Zygmund (MZ) property: $\exists\ \eta \in [0,1)$ such that

$$\left|\sum_{j=1}^{m} w_j \chi(x_j)^2 - \int_\Omega \chi^2 \mathrm{d}\omega_d\right| \leq \eta \int_\Omega \chi^2 \mathrm{d}\omega_d \quad \forall \chi \in \mathbb{P}_n.$$

What if **relaxing** $2n$ to, say, $n+k$ with $0 < k \leq n$?

Theorem (An and W. 2022)

Assume the quadrature rule has exactness degree $n+k$ and **satisfies the MZ property**. Then for any $f \in C(\Omega)$:

- $\mathcal{L}_n \chi = \chi$ for any $\chi \in \mathbb{P}_k$;
- $\|\mathcal{L}_n f\|_2 \leq \dfrac{V^{1/2}}{\sqrt{1-\eta}} \|f\|_\infty$;
- $\|\mathcal{L}_n f - f\|_2 \leq \left(\dfrac{1}{\sqrt{1-\eta}} + 1\right) V^{1/2} E_k(f)$.

Remark: If the quadrature rule has exactness degree $2n$ (or $k = n$), then

$$\eta = 0 \implies \text{Sloan's original results.}$$

Why Marcinkiewicz–Zygmund?

- (with exactness degree of $2n$) The key observation for the stability:

$$\|\mathcal{L}_n f\|_2^2 + \underbrace{\langle f - \mathcal{L}_n f, f - \mathcal{L}_n f\rangle_m}_{\geq 0 \text{ (all } w_j > 0)} = \langle f, f\rangle_m = \sum_{j=1}^{m} w_j f(x_j)^2 \leq V\|f\|_\infty^2$$

- (with exactness degree being $n+k$, $0 < k \leq n$) We can only derive:

$$\|\mathcal{L}_n f\|_2^2 + \underbrace{\langle f - \mathcal{L}_n f, f - \mathcal{L}_n f\rangle_m + \sigma_{m,n,f}}_{\geq 0?} = \langle f, f\rangle_m;$$

where $\sigma_{n,k,f} = \langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle - \langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle_m$.

Note that $\mathcal{L}_n f - \mathcal{L}_k f \in \mathbb{P}_n$, the MZ property implies

$$|\sigma_{n,k,f}| \leq \eta\langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle \leq \eta\|\mathcal{L}_n f\|_2^2.$$

Numerical results on $[-1,1]$

Let p_ℓ be normalized (orthonormal) Legendre polynomials on $[-1,1]$ with $d_n = \dim \mathbb{P}_n = n+1$.

- Gauss–Legendre quadrature
- Clenshaw–Curtis quadrature
- DeVore, Foucart, Petrova, and Wojtaszczy (2019):

$$\min_{w_1, w_2, \ldots, w_m} \sum_{j=1}^{m} |w_j| \ \text{ s.t. } \ \sum_{j=1}^{m} w_j g(x_j) = \int_{-1}^{1} g(x)\,dx \quad \forall g \in \mathbb{P}_{n+k}.$$

Based on this DFPW optimization problem, we generate quadrature rules in equispaced points with certain exactness degrees.

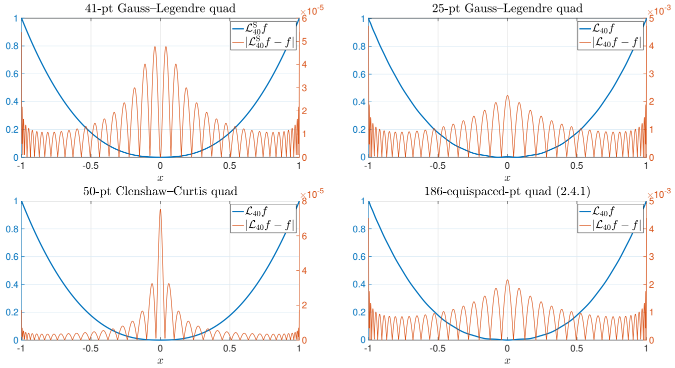


Figure: Hyperinterpolants $\mathcal{L}_{40}^{\mathrm{S}}f$ and $\mathcal{L}_{40}f$ of $f(x) = |x|^{5/2}$, constructed by various quadrature rules. Except for the one on the top left, all other quadrature rules have exactness degree 49.

❑ To our best knowledge, the connection between the Clenshaw–Curtis quadrature and the performance of hyperinterpolation has not been established.

Numerical results on $\mathbb{S}^2$

Let p_ℓ be spherical harmonics on $\mathbb{S}^2$ with $d_n = \dim \mathbb{P}_n = (n+1)^2$

Definition (Delsarte, Goethals, and Seidel 1977)

A point set $\{x_1, \ldots, x_m\} \subset \mathbb{S}^2$ is said to be a **spherical t-design** if it satisfies

$$\frac{1}{m}\sum_{j=1}^{m} g(x_j) = \frac{1}{4\pi}\int_{\mathbb{S}^2} g \mathrm{d}\omega \quad \forall g \in \mathbb{P}_t.$$

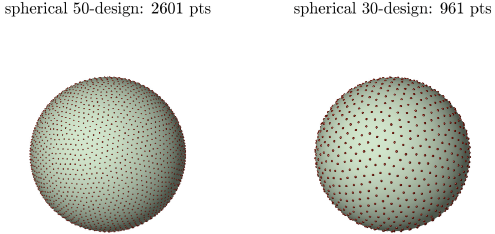


Figure: Spherical 50- and 30-designs, generated by the optimization method proposed by An, Chen, Sloan, and Womersley (2010).

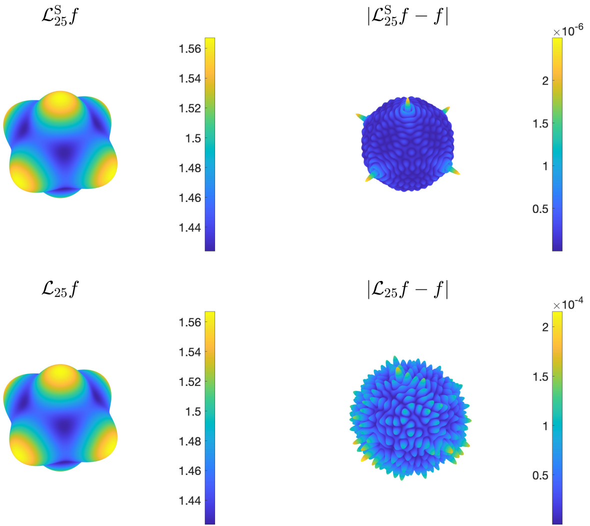


Figure: Hyperinterpolants $\mathcal{L}_{25}^{\mathrm{S}} f$ and $\mathcal{L}_{25} f$ of a Wendland function, constructed by spherical t-designs with $t = 50$ (upper row) and 30 (lower row).

What if totally discarding quadrature exactness?

A case study on **spheres**: The "polynomial" space $\mathbb{P}_n(\mathbb{S}^d)$ is the span of spherical harmonics $\{Y_{\ell,k}: \ \ell = 0, 1, \ldots, n, \ k = 1, 2, \ldots, Z(d,\ell)\}$; $\mathbb{P}_n(\mathbb{S}^d)$ is also a **reproducing kernel Hilbert space** with the reproducing kernel

$$G_n(x,y) = \sum_{\ell=0}^{n} \sum_{k=1}^{Z(d,\ell)} Y_{\ell,k}(x) Y_{\ell,k}(y)$$

in the sense that $\langle \chi, G(\cdot, x) \rangle = \chi(x)$ for all $\chi \in \mathbb{P}_n(\mathbb{S}^d)$.

For hyperinterpolation w/o quadrature exactness:

$$\mathcal{L}_n f(x) = \sum_{\ell=0}^{n} \sum_{k=1}^{Z(d,\ell)} \left(\sum_{j=1}^{m} w_j f(x_j) Y_{\ell,k}(x_j) \right) Y_{\ell,k}(x) = \sum_{j=1}^{m} w_j f(x_j) G_n(x, x_j)$$

$$\langle \mathcal{L}_n \chi, \chi \rangle = \left\langle \textstyle\sum_{j=1}^m w_j \chi(x_j) G_n(x, x_j), \chi(x) \right\rangle = \textstyle\sum_{j=1}^m w_j \chi(x_j)^2$$

Theorem (An and W. 2024)

Assume the quadrature rule **satisfies the MZ property** only. Then for any $f \in C(\Omega)$:

- Not a projection operator anymore;
- $\|\mathcal{L}_n f\|_{L^2} \leq \sqrt{1+\eta}\left(\sum_{j=1}^{m} w_j\right)^{1/2} \|f\|_\infty$;
- $$\|\mathcal{L}_n f - f\|_{L^2} \leq \left(\sqrt{1+\eta}\left(\sum_{j=1}^{m} w_j\right)^{1/2} + |\mathbb{S}^d|^{1/2}\right) E_n(f) + \sqrt{\eta^2+4\eta}\|\chi^*\|_{L^2}.$$

Note: If the quadrature rule has exactness degree at least **1**, then

$$\sum_{j=1}^{m} w_j = \int_{\mathbb{S}^d} 1 \mathrm{d}\omega_d = |\mathbb{S}^d|.$$

Error bound investigated numerically

- The error bound is controlled by n and m
- Le Gia & Mhaskar (random points; $m \geq \bar{c} n^d \log n / \eta^2$ on $\mathbb{S}^d$)
  $\Rightarrow \eta$ has a lower bound order $\sqrt{n^2 \log n / m}$
  $\Rightarrow \sqrt{\eta^2 + 4\eta} \|\chi^*\|_{L^2}$ has a lower bound of order $m^{-1/4}$ w.r.t. m, and it increases as n enlarges

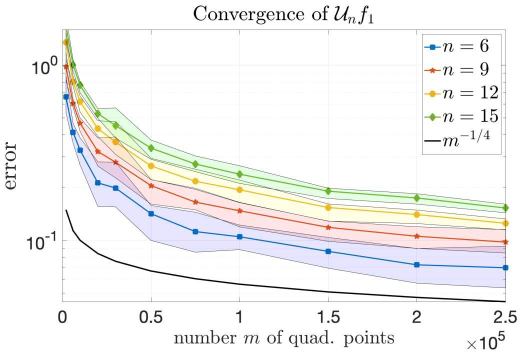


Figure: Approximating $f_1(x) = (x_1 + x_2 + x_3)^2$.

- $f_2(x_1, x_2, x_3) := |x_1 + x_2 + x_3| + \sin^2(1 + |x_1 + x_2 + x_3|)$
- The Franke function for the sphere

$$\begin{aligned} f_3(x_1, x_2, x_3) := &0.75 \exp(-((9x_1 - 2)^2)/4 - ((9x_2 - 2)^2)/4 - ((9x_3 - 2)^2)/4) \\ &+0.75 \exp(-((9x_1 + 1)^2)/49 - ((9x_2 + 1))/10 - ((9x_3 + 1))/10) \\ &+0.5 \exp(-((9x_1 - 7)^2)/4 - ((9x_2 - 3)^2)/4 - ((9x_3 - 5)^2)/4) \\ &-0.2 \exp(-((9x_1 - 4)^2) - ((9x_2 - 7)^2) - ((9x_3 - 5)^2)) \in C^\infty(\mathbb{S}^2) \end{aligned}$$

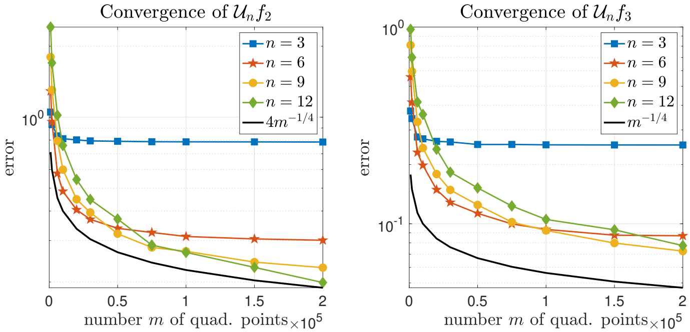


Figure: Approximating f_2 and f_3 (the notation $\mathcal{U}_n$ stands for hyperinterpolation, as adopted in our publication).

References

The original taste

- I. H. Sloan (1995). Polynomial interpolation and hyperinterpolation over general regions. *Journal of Approximation Theory*, 83(2), 238–254.

Our contributions included in this talk

- C. An & W. (2022). On the quadrature exactness in hyperinterpolation. *BIT Numerical Mathematics*, 62(4), 1899–1919.
- C. An & W. (2024). Bypassing the quadrature exactness assumption of hyperinterpolation on the sphere. *Journal of Complexity*, 80, 101789. (More numerical experiments involving equal area points, Fekete points, and minimal energy points are available in this work)

Further applications of "hyperinterpolation + MZ"

- W. & X. Yuan. Breaking quadrature exactness: A spectral method for the Allen–Cahn equation on spheres. arXiv:2305.04820.
- C. An & W. Spherical configurations and quadrature methods for integral equations of the second kind. arXiv:2408.14392.

Thanks for your attention.

Photo taken from the State Botanical Garden of Georgia, Athens, GA.