Hyperinterpolation, Marcinkiewicz–Zygmund property, and their use for spectral methods†

†based on joint works with Congpei An (SWUFE) and Xiaoming Yuan (HKU)

Hao-Ning Wu
The University of Hong Kong
Jun 4, 2024

International Conference on Analysis and Applications 2024
in honor of Roderick S. C. Wong's 80th birthday
City University of Hong Kong

Polynomial approximation

☞ For $f \in C(\Omega)$, find an approximant $p = \sum_{\ell=1}^{d_n} c_\ell p_\ell \in \mathbb{P}_n$:

- ❑ $\Omega \subset \mathbb{R}^d$: bounded, closed subset of $\mathbb{R}^d$ or compact manifold with finite measure w.r.t a given (positive) measure $\mathrm{d}\omega$, i.e., $\int_\Omega \mathrm{d}\omega = V$.
- ❑ $\mathbb{P}_n$: space of polynomials of degree $\leq n$ over Ω
- ❑ $\{p_1, p_2, \dots, p_{d_n}\}$: orthonormal basis of $\mathbb{P}_n$ with dim. $d_n := \dim \mathbb{P}_n$

☞ Famous Methods:

- ❑ **Polynomial interpolation**: given points $\{x_j\}_{j=1}^{d_n}$, find p such that

$$f(x_j) = p(x_j) = \sum_{\ell=1}^{d_n} c_\ell p_\ell(x_j), \quad j = 1, \dots, d_n$$

  \- complicated and even problematic in multivariate cases
- ❑ **Orthogonal projection**: defined as

$$\mathcal{P}_n f := \sum_{\ell=1}^{d_n} \langle f, p_\ell \rangle p_\ell,$$

  where $\langle f, g \rangle = \int_\Omega fg \mathrm{d}\omega$

  \- non-implementable on computers

Ian H. Sloan (in the early 1990s): Does the interpolation of functions on $\mathbb{S}^1$ have properties as good as orthogonal projection?

✍ Sloan ('95 JAT)

- on $\mathbb{S}^1$: Yes.
- on $\mathbb{S}^d$ ($d \geq 2$) and most high-dim regions: remaining **Problematic** to this day!
- Using more points than interpolation? $\rightarrow$ **hyper**interpolation

The **hyperinterpolation** of $f \in C(\Omega)$ onto $\mathbb{P}_n$ is defined as

$$\mathcal{L}_n f := \sum_{\ell=1}^{d_n} \langle f, p_\ell \rangle_m p_\ell,$$

where $\langle f, g \rangle_m := \sum_{j=1}^{m} w_j f(x_j) g(x_j)$ with all $w_j > 0$.

- $\mathcal{L}_n f$ is a discretized version of the **orthogonal projection** $\mathcal{P}_n f$.
- $\mathcal{L}_n f$ reduces to **interpolation** if the quadrature rule is d_n-point with exactness degree exceeding $2n$.

The quadrature rule $\sum_{j=1}^{m} w_j g(x_j) \approx \int_\Omega g \mathrm{d}\omega$ is said to have **exactness degree** $2n$ if

$$\sum_{j=1}^{m} w_j g(x_j) = \int_\Omega g \mathrm{d}\omega \quad \forall g \in \mathbb{P}_{2n}.$$

Caveat: Such quadrature rules (d_n-point with exactness degree at least $2n$) only exist on a few low-dimensional Ω, such as $[-1,1]$, $[-1,1]^2$, and $\mathbb{S}^1$; and they are not available on $[-1,1]^d$ ($d \geq 3$) or $\mathbb{S}^d$ ($d \geq 2$).

In higher dimensions, more quadrature points (than d_n) are necessary for exactness degree $2n$

Theorem (Sloan 1995)

Assume the quadrature rule has exactness degree $2n$. Then for any $f \in C(\Omega)$, its hyperinterpolant $\mathcal{L}_n f$ satisfies:

- $\mathcal{L}_n \chi = \chi$ for any $\chi \in \mathbb{P}_n$;
- $\|\mathcal{L}_n f\|_2 \leq V^{1/2} \|f\|_\infty$;
- $\|\mathcal{L}_n f - f\|_2 \leq 2V^{1/2} E_n(f)$.

Here $V = |\Omega|$ and $E_n(f) := \inf_{\chi \in \mathbb{P}_n} \|f - \chi\|_\infty$.

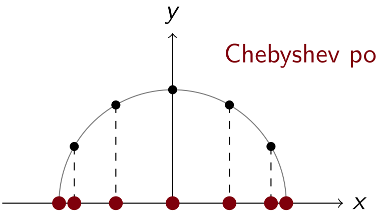
➥ The interpolation of functions on $\mathbb{S}^1$ has properties as good as orthogonal projection ✓

➥ That on $\mathbb{S}^2$ or higher dimensional spheres ?

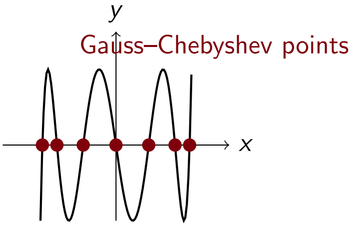
The theory relies on quadrature exactness of degree at least $2n$:

$$\sum_{j=1}^{m} w_j g(x_j) = \int_\Omega g \, d\omega \quad \forall g \in \mathbb{P}_{2n}.$$

On quadrature exactness



Clenshaw–Curtis quad (1960)
$n+1$ points $\to$ n exactness degree



Gauss–Chebyshev quad (19th century)
$n+1$ points $\to$ $2n+1$ exactness degree

✍ Trefethen ('08 SIREV): entered the complex plane and demonstrated for most functions, the Clenshaw–Curtis and Gauss quadrature rules have comparable accuracy

✍ Trefethen ('22 SIREV): numerical integral is an analysis topic, while quadrature exactness is an algebraic matter

Our solution: Marcinkiewicz–Zygmund

✍ Marcinkiewicz and Zygmund (1937): There exists $\eta \in [0,1)$ such that

$$(1-\eta)\int_\Omega \chi^2 \mathrm{d}\omega_d \le \sum_{j=1}^m w_j \chi(x_j)^2 \le (1+\eta)\int_\Omega \chi^2 \mathrm{d}\omega_d \quad \forall \chi \in \mathbb{P}_n.$$

- MZ on spheres: Mhaskar, Narcowich, & Ward (2001)
- MZ on compact manifolds: Filbir & Mhaskar (2011)
- MZ on multivariate domains other than compact manifolds (balls, polytopes, cones, spherical sectors, etc.): De Marchi & Kroó (2018)

In particular, $[h_{\mathcal{X}_m} := \max_{x\in\mathbb{S}^{d-1}} \min_{x_j\in\mathcal{X}_m} \mathrm{dist}(x, x_j)]$

- MZ on compact manifolds holds if $n \lesssim \eta / h_{\mathcal{X}_m}$, where $h_{\mathcal{X}_m}$ is the mesh norm of $\{x_j\}_{j=1}^m \Rightarrow$ **Scattered data**
- Le Gia and Mhaskar (2009): If $\{x_j\}$ are i.i.d drawn from the distribution ω_d, then there exists a constant $\bar{c} := \bar{c}(\gamma)$ such that MZ holds on $\mathbb{S}^d$ with probability $\ge 1 - \bar{c}N^{-\gamma}$ on the condition $m \ge \bar{c}N^d \log N/\eta^2 \Rightarrow$ **Random data** and **learning theory**

First-stage solution

Marcinkiewicz–Zygmund (MZ) property (equiv.): $\exists\, \eta \in [0,1)$ such that

$$\left|\sum_{j=1}^{m} w_j \chi(x_j)^2 - \int_\Omega \chi^2 \mathrm{d}\omega_d\right| \leq \eta \int_\Omega \chi^2 \mathrm{d}\omega_d \quad \forall \chi \in \mathbb{P}_n.$$

What if **relaxing** $2n$ to, say, $n+k$ with $0 < k \leq n$?

Theorem (An and W. 2022)

Assume the quadrature rule has exactness degree $n+k$ and **satisfies the MZ property**. Then for any $f \in C(\Omega)$:

- $\mathcal{L}_n \chi = \chi$ for any $\chi \in \mathbb{P}_k$;
- $\|\mathcal{L}_n f\|_2 \leq \dfrac{V^{1/2}}{\sqrt{1-\eta}} \|f\|_\infty$;
- $\|\mathcal{L}_n f - f\|_2 \leq \left(\dfrac{1}{\sqrt{1-\eta}} + 1\right) V^{1/2} E_k(f)$.

Remark: If the quadrature rule has exactness degree $2n$ (or $k = n$), then $\eta = 0 \implies$ Sloan's original results.

Why Marcinkiewicz–Zygmund?

- (with exactness degree of $2n$) The key observation for the stability of $\mathcal{L}_n f$:

$$\|\mathcal{L}_n f\|_2^2 + \underbrace{\langle f - \mathcal{L}_n f, f - \mathcal{L}_n f\rangle_m}_{\geq 0 \text{ (all } w_j > 0)} = \langle f, f\rangle_m = \sum_{j=1}^{m} w_j f(x_j)^2 \leq V\|f\|_\infty^2$$

- (with exactness degree being $n+k$, $0 < k \leq n$) We can only derive:

$$\|\mathcal{L}_n f\|_2^2 + \underbrace{\langle f - \mathcal{L}_n f, f - \mathcal{L}_n f\rangle_m + \sigma_{m,n,f}}_{\geq 0?} = \langle f, f\rangle_m;$$

where $\sigma_{n,k,f} = \langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle - \langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle_m$.

➥ Note that $\mathcal{L}_n f - \mathcal{L}_k f \in \mathbb{P}_n$, the MZ property implies

$$|\sigma_{n,k,f}| \leq \eta\langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle \leq \eta\|\mathcal{L}_n f\|_2^2.$$

Let p_ℓ be spherical harmonics on $\mathbb{S}^2$ with $d_n = \dim \mathbb{P}_n = (n+1)^2$

Definition (Delsarte, Goethals, and Seidel 1977)

A point set $\{x_1, \ldots, x_m\} \subset \mathbb{S}^2$ is said to be a **spherical t-design** if it satisfies

$$\frac{1}{m}\sum_{j=1}^{m} g(x_j) = \frac{1}{4\pi}\int_{\mathbb{S}^2} g \mathrm{d}\omega \quad \forall g \in \mathbb{P}_t.$$

spherical 50-design: 2601 pts　　　　spherical 30-design: 961 pts

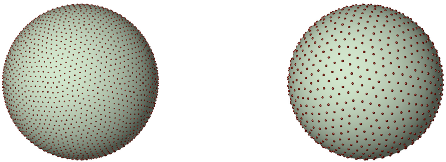


Figure: Spherical 50- and 30-designs, generated by the optimization method proposed by An, Chen, Sloan, and Womersley (2010).

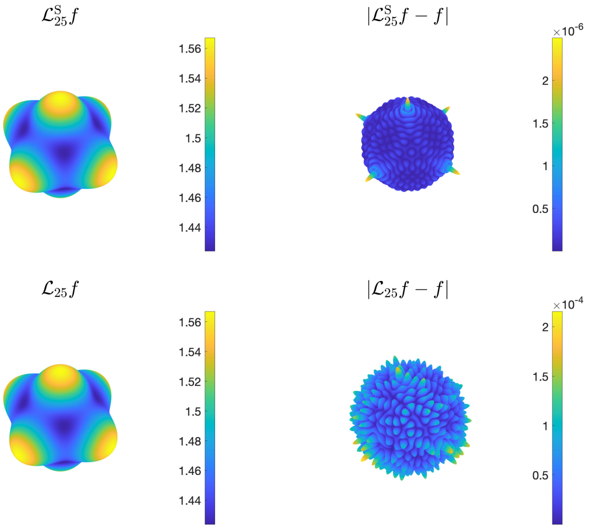


Figure: Hyperinterpolants $\mathcal{L}_{25}^{\mathsf{S}} f$ and $\mathcal{L}_{25} f$ of a Wendland function, constructed by spherical t-designs with $t = 50$ (upper row) and 30 (lower row).

What if totally discarding quadrature exactness?

A case study on **spheres**: The "polynomial" space $\mathbb{P}_n(\mathbb{S}^d)$ is the span of spherical harmonics $\{Y_{\ell,k}: \ \ell = 0, 1, \ldots, n, \ k = 1, 2, \ldots, Z(d, \ell)\}$; $\mathbb{P}_n(\mathbb{S}^d)$ is also a **reproducing kernel Hilbert space** with the reproducing kernel

$$G_n(x, y) = \sum_{\ell=0}^{n} \sum_{k=1}^{Z(d,\ell)} Y_{\ell,k}(x) Y_{\ell,k}(y)$$

in the sense that $\langle \chi, G(\cdot, x) \rangle = \chi(x)$ for all $\chi \in \mathbb{P}_n(\mathbb{S}^d)$.

For hyperinterpolation (denoted by $U_n f$ w/o quadrature exactness):

$$\mathcal{U}_n f(x) = \sum_{\ell=0}^{n} \sum_{k=1}^{Z(d,\ell)} \left(\sum_{j=1}^{m} w_j f(x_j) Y_{\ell,k}(x_j) \right) Y_{\ell,k}(x) = \sum_{j=1}^{m} w_j f(x_j) G_n(x, x_j)$$

$$\langle \mathcal{U}_n \chi, \chi \rangle = \left\langle \sum_{j=1}^m w_j \chi(x_j) G_n(x, x_j), \chi(x) \right\rangle = \sum_{j=1}^m w_j \chi(x_j)^2$$

Theorem (An and W. 2024)

Assume the quadrature rule **satisfies the MZ property**. Then for any $f \in C(\Omega)$:

- $\|\mathcal{U}_n f\|_{L^2} \leq \sqrt{1+\eta}\left(\sum_{j=1}^{m} w_j\right)^{1/2} \|f\|_\infty$;
- $$\|\mathcal{U}_n f - f\|_{L^2} \leq \left(\sqrt{1+\eta}\left(\sum_{j=1}^{m} w_j\right)^{1/2} + |\mathbb{S}^d|^{1/2}\right) E_n(f) + \sqrt{\eta^2+4\eta}\|\chi^*\|_{L^2},$$

where $\mathcal{U}_n$ stands for hyperinterpolation without quadrature exactness.

Note: If the quadrature rule has exactness degree at least **1**, then

$$\sum_{j=1}^{m} w_j = \int_{\mathbb{S}^d} 1 \mathrm{d}\omega_d = |\mathbb{S}^d|.$$

Error bound investigated numerically

- The error bound is controlled by n and m
- Le Gia & Mhaskar (random points)
  $\Rightarrow \eta$ has a lower bound order $\sqrt{n^2 \log n / m}$
  $\Rightarrow \sqrt{\eta^2 + 4\eta}\|\chi^*\|_{L^2}$ has a lower bound of order $m^{-1/4}$ w.r.t. m, and it increases as n enlarges

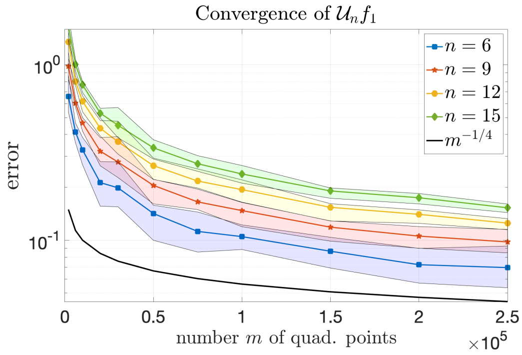


Figure: Approximating $f_1(x) = (x_1 + x_2 + x_3)^2 \in \mathbb{P}_6(\mathbb{S}^2)$.

- $f_2(x_1, x_2, x_3) := |x_1 + x_2 + x_3| + \sin^2(1 + |x_1 + x_2 + x_3|)$
- The Franke function for the sphere

$$
\begin{aligned}
f_3(x_1, x_2, x_3) := & 0.75 \exp(-((9x_1 - 2)^2)/4 - ((9x_2 - 2)^2)/4 - ((9x_3 - 2)^2)/4) \\
& + 0.75 \exp(-((9x_1 + 1)^2)/49 - ((9x_2 + 1))/10 - ((9x_3 + 1))/10) \\
& + 0.5 \exp(-((9x_1 - 7)^2)/4 - ((9x_2 - 3)^2)/4 - ((9x_3 - 5)^2)/4) \\
& - 0.2 \exp(-((9x_1 - 4)^2) - ((9x_2 - 7)^2) - ((9x_3 - 5)^2)) \in C^\infty(\mathbb{S}^2)
\end{aligned}
$$

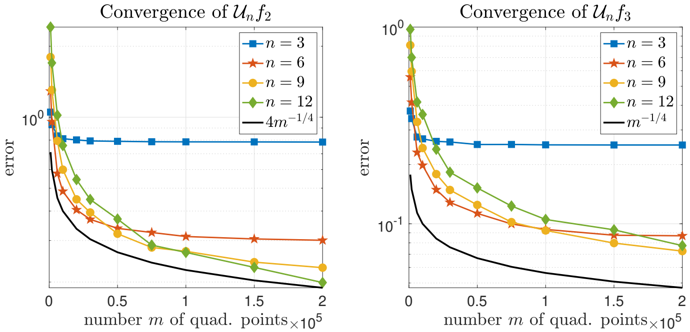


Figure: Approximating f_2 and f_3.

Applications to nonlinear partial differential equations (PDEs)

To compute smooth solutions of **semi-linear PDEs** on $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ with dimension $d \geq 3$ of the form

$$u_t = \boldsymbol{L}u + \boldsymbol{N}(u), \quad u(0,x) = u_0(x),$$

where $\boldsymbol{L}$ is a constant-coefficient linear differential operator, and $\boldsymbol{N}$ is a constant-coefficient nonlinear differential (or non-differential) operator of lower order.

Example: **Allen–Cahn equation**

$$u_t = \nu^2 \Delta u - F'(u), \quad u(0,x) = u_0(x),$$

where $F'(u) = f(u) = u^3 - u$.

✍ W. & Yuan (23' Preprint): Our idea in a nutshell: **linearizing** the nonlinear part $\boldsymbol{N}(u)$ by hyperinterpolation:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = \nu^2 \Delta u^{n+1} - \mathcal{L}_N \left((u^n)^3 - u^n \right), \quad n \geq 0, \\ u^0 = \mathcal{L}_N u_0 \end{cases}$$

where $\tau > 0$ is the time step.

For each time iteration: using the property $-\Delta Y_{\ell,k} = \ell(\ell + d - 2) Y_{\ell,k}$ of spherical Laplace–Beltrami operator, only solving a **linear system**

We have analyzed the L^∞ stability and maximum principle for the scheme. In our analysis, we assume $\eta = cN^{-s}$ for $s > (d-1)/2$ and some constant $c > 0$.

- **Scattered data**: MZ holds if $N \lesssim \eta / h_{\mathcal{X}_m}$
- **Random data**: MZ holds with probability exceeding $1 - \bar{c}N^{-\gamma}$ on the condition $N^{d-1} \log N \lesssim \eta^2 m$

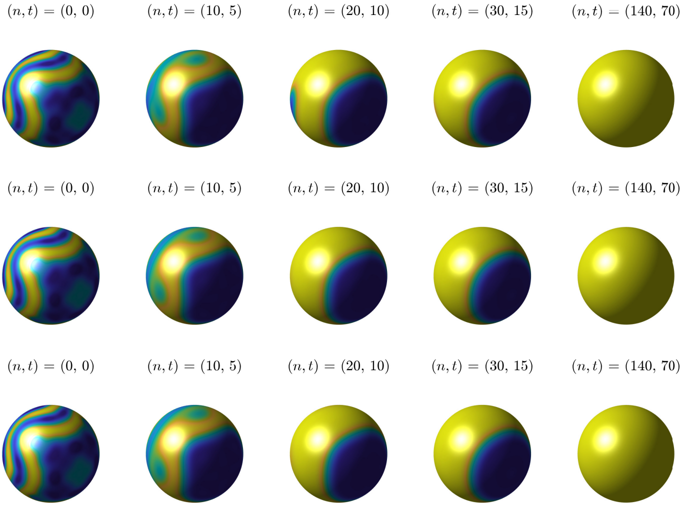


Figure: Numerical solution to the Allen–Cahn equation with $\nu = 0.1$ and initial condition $u(0, x, y, z) = \cos(\cosh(5xz) - 10y)$ using our scheme with $\tau = 0.5$, $N = 15$, and different quadrature points. From top row to bottom row: $m = \lfloor 120N^2 \ln N \rfloor = 73{,}117$ **random points**; $m = (2N+1)^2 = 961$ **equal area points**; and $m = 961$ **spherical $2N$-designs**.

Happy Birthday, Professor Wong!