Hyperinterpolation, Marcinkiewicz–Zygmund property, and their use for integral equations†

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Polynomial approximation

☞ For $f \in C(\Omega)$, find an approximant $p = \sum_{\ell=1}^{d_n} c_\ell p_\ell \in \mathbb{P}_n$:

- $\Omega \subset \mathbb{R}^d$: bounded, closed subset of $\mathbb{R}^d$ or compact manifold with finite measure w.r.t a given (positive) measure $\mathrm{d}\omega$, i.e., $\int_\Omega \mathrm{d}\omega = V$.
- $\mathbb{P}_n$: space of polynomials of degree $\leq n$ over Ω
- $\{p_1, p_2, \ldots, p_{d_n}\}$: orthonormal basis of $\mathbb{P}_n$ with dim. $d_n := \dim \mathbb{P}_n$

☞ Famous Methods:

- **Polynomial interpolation**: given points $\{x_j\}_{j=1}^{d_n}$, find p such that

$$\boxed{f(x_j) = p(x_j)} = \sum_{\ell=1}^{d_n} c_\ell p_\ell(x_j), \quad j = 1, \ldots, d_n$$

  - complicated and even problematic in multivariate cases

- **Orthogonal projection**: defined as

$$\boxed{\mathcal{P}_n f := \sum_{\ell=1}^{d_n} \langle f, p_\ell \rangle p_\ell,}$$

  where $\langle f, g \rangle = \int_\Omega f g \mathrm{d}\omega$

  - non-implementable on computers

Hyperinterpolation

Ian H. Sloan (in the early 1990s): Does the interpolation of functions on $\mathbb{S}^1$ have properties as good as orthogonal projection?

✍ Sloan ('95 JAT)

- on $\mathbb{S}^1$: Yes.
- on $\mathbb{S}^d$ ($d \geq 2$) and most high-dim regions: remaining **Problematic** to this day!
- Using more points than interpolation? $\rightarrow$ **hyper**interpolation

The **hyperinterpolation** of $f \in C(\Omega)$ onto $\mathbb{P}_n$ is defined as

$$\mathcal{L}_n f := \sum_{\ell=1}^{d_n} \langle f, p_\ell \rangle_m p_\ell,$$

where $\langle f, g \rangle_m := \sum_{j=1}^{m} w_j f(x_j) g(x_j)$ with all $w_j > 0$.

- $\mathcal{L}_n f$ is a discretized version of the **orthogonal projection** $\mathcal{P}_n f$.
- $\mathcal{L}_n f$ reduces to **interpolation** if the quadrature rule is d_n-point with exactness degree exceeding $2n$.

The quadrature rule $\sum_{j=1}^{m} w_j g(x_j) \approx \int_\Omega g \mathrm{d}\omega$ is said to have **exactness degree** $2n$ if

$$\sum_{j=1}^{m} w_j g(x_j) = \int_\Omega g \mathrm{d}\omega \quad \forall g \in \mathbb{P}_{2n}.$$

Caveat: Such quadrature rules (d_n-point with exactness degree at least $2n$) only exist on a few low-dimensional Ω, such as $[-1,1]$, $[-1,1]^2$, and $\mathbb{S}^1$; and they are not available on $[-1,1]^d$ ($d \geq 3$) or $\mathbb{S}^d$ ($d \geq 2$).

In higher dimensions, more quadrature points (than d_n) are necessary for exactness degree $2n$

Theorem (Sloan '95 JAT)

Assume the quadrature rule has exactness degree $2n$. Then for any $f \in C(\Omega)$, its hyperinterpolant $\mathcal{L}_n f$ satisfies:

- $\mathcal{L}_n \chi = \chi$ for any $\chi \in \mathbb{P}_n$;
- $\|\mathcal{L}_n f\|_2 \leq V^{1/2} \|f\|_\infty$;
- $\|\mathcal{L}_n f - f\|_2 \leq 2V^{1/2} E_n(f)$.

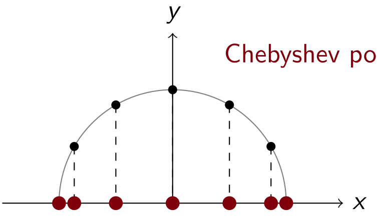
Here $V = |\Omega|$ and $E_n(f) := \inf_{\chi \in \mathbb{P}_n} \|f - \chi\|_\infty$.

☞ The interpolation of functions on $\mathbb{S}^1$ has properties as good as orthogonal projection ✓
☞ That on $\mathbb{S}^2$ or higher dimensional spheres ?

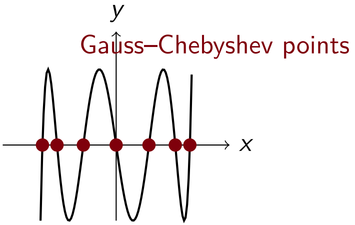
Caveat: The theorem relies on quad. exactness of degree at least $2n$:

$$\sum_{j=1}^{m} w_j g(x_j) = \int_\Omega g \mathrm{d}\omega \quad \forall g \in \mathbb{P}_{2n}.$$

On quadrature exactness



Clenshaw–Curtis quad (1960)
$n+1$ points $\to$ n exactness degree



Gauss–Chebyshev quad (19th century)
$n+1$ points $\to$ $2n+1$ exactness degree

✍ Trefethen ('08 SIREV): entered the complex plane and demonstrated for most functions, the Clenshaw–Curtis and Gauss quadrature rules have comparable accuracy

✍ Trefethen ('22 SIREV): numerical integral is an analysis topic, while quadrature exactness is an algebraic matter

Our solution: Marcinkiewicz–Zygmund

✍ Marcinkiewicz and Zygmund (1937): There exists $\eta \in [0,1)$ such that

$$(1-\eta)\int_\Omega \chi^2 \mathrm{d}\omega_d \le \sum_{j=1}^m w_j \chi(x_j)^2 \le (1+\eta)\int_\Omega \chi^2 \mathrm{d}\omega_d \quad \forall \chi \in \mathbb{P}_n.$$

- MZ on spheres: Mhaskar, Narcowich, & Ward (2001)
- MZ on compact manifolds: Filbir & Mhaskar (2011)
- MZ on multivariate domains other than compact manifolds (balls, polytopes, cones, spherical sectors, etc.): De Marchi & Kroó (2018)

In particular, $[h_{\mathcal{X}_m} := \max_{x\in\mathbb{S}^{d-1}} \min_{x_j\in\mathcal{X}_m} \mathrm{dist}(x, x_j)]$

- MZ on compact manifolds holds if $n \lesssim \eta / h_{\mathcal{X}_m}$, where $h_{\mathcal{X}_m}$ is the mesh norm of $\{x_j\}_{j=1}^m \Rightarrow$ Scattered data
- Le Gia and Mhaskar (2009): If $\{x_j\}$ are i.i.d drawn from the distribution ω_d, then there exists a constant $\bar{c} := \bar{c}(\gamma)$ such that MZ holds on $\mathbb{S}^d$ with probability $\ge 1 - \bar{c}N^{-\gamma}$ on the condition $m \ge \bar{c}N^d \log N/\eta^2 \Rightarrow$ Random data and learning theory

First-stage solution

Marcinkiewicz–Zygmund (MZ) property: $\exists\ \eta \in [0,1)$ such that

$$\left|\sum_{j=1}^{m} w_j \chi(x_j)^2 - \int_\Omega \chi^2 \mathrm{d}\omega_d\right| \leq \eta \int_\Omega \chi^2 \mathrm{d}\omega_d \quad \forall \chi \in \mathbb{P}_n.$$

What if **relaxing** $2n$ to, say, $n+k$ with $0 < k \leq n$?

Theorem (An and W. '22 BIT)

Assume the quadrature rule has exactness degree $n+k$ and **satisfies the MZ property**. Then for any $f \in C(\Omega)$:

- $\mathcal{L}_n \chi = \chi$ for any $\chi \in \mathbb{P}_k$;
- $\|\mathcal{L}_n f\|_2 \leq \dfrac{V^{1/2}}{\sqrt{1-\eta}} \|f\|_\infty$;
- $\|\mathcal{L}_n f - f\|_2 \leq \left(\dfrac{1}{\sqrt{1-\eta}} + 1\right) V^{1/2} E_k(f)$.

Remark: If the quadrature rule has exactness degree $2n$ (or $k = n$), then $\eta = 0 \implies$ Sloan's original results.

Why Marcinkiewicz–Zygmund?

- (with exactness degree of $2n$) The key observation for the stability:

$$\|\mathcal{L}_n f\|_2^2 + \underbrace{\langle f - \mathcal{L}_n f, f - \mathcal{L}_n f\rangle_m}_{\geq 0 \text{ (all } w_j > 0)} = \langle f, f\rangle_m = \sum_{j=1}^{m} w_j f(x_j)^2 \leq V\|f\|_\infty^2$$

- (with exactness degree being $n+k$, $0 < k \leq n$) We can only derive:

$$\|\mathcal{L}_n f\|_2^2 + \underbrace{\langle f - \mathcal{L}_n f, f - \mathcal{L}_n f\rangle_m + \sigma_{m,n,f}}_{\geq 0?} = \langle f, f\rangle_m;$$

where $\sigma_{n,k,f} = \langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle - \langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle_m$.

➥ Note that $\mathcal{L}_n f - \mathcal{L}_k f \in \mathbb{P}_n$, the MZ property implies

$$|\sigma_{n,k,f}| \leq \eta \langle \mathcal{L}_n f - \mathcal{L}_k f, \mathcal{L}_n f - \mathcal{L}_k f\rangle \leq \eta \|\mathcal{L}_n f\|_2^2.$$

Numerical results on $\mathbb{S}^2$

Let p_ℓ be spherical harmonics on $\mathbb{S}^2$ with $d_n = \dim \mathbb{P}_n = (n+1)^2$

Definition (Delsarte, Goethals, and Seidel 1977)

A point set $\{x_1, \ldots, x_m\} \subset \mathbb{S}^2$ is said to be a **spherical t-design** if it satisfies

$$\frac{1}{m}\sum_{j=1}^{m} g(x_j) = \frac{1}{4\pi}\int_{\mathbb{S}^2} g \mathrm{d}\omega \quad \forall g \in \mathbb{P}_t.$$

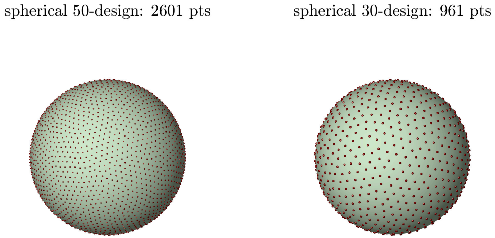


Figure: Spherical 50- and 30-designs, generated by the optimization method proposed by An, Chen, Sloan, and Womersley (2010).

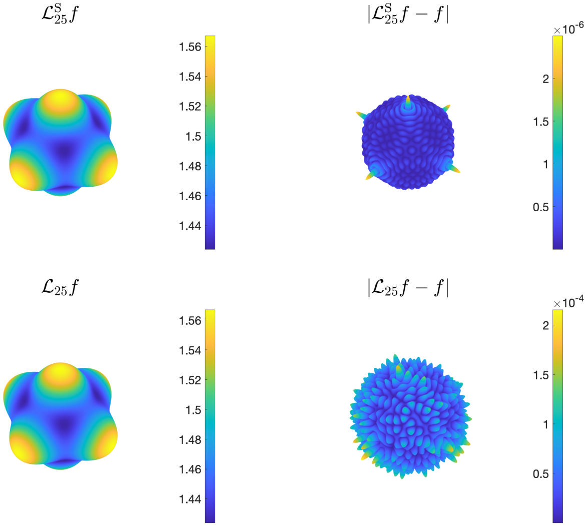


Figure: Hyperinterpolants $\mathcal{L}_{25}^{\mathrm{S}} f$ and $\mathcal{L}_{25} f$ of a Wendland function, constructed by spherical t-designs with $t = 50$ (upper row) and 30 (lower row).

What if totally discarding quadrature exactness?

A case study on **spheres**: The "polynomial" space $\mathbb{P}_n(\mathbb{S}^d)$ is the span of spherical harmonics $\{Y_{\ell,k}: \ \ell = 0, 1, \ldots, n, \ k = 1, 2, \ldots, Z(d, \ell)\}$; $\mathbb{P}_n(\mathbb{S}^d)$ is also a **reproducing kernel Hilbert space** with the reproducing kernel

$$G_n(x, y) = \sum_{\ell=0}^{n} \sum_{k=1}^{Z(d,\ell)} Y_{\ell,k}(x) Y_{\ell,k}(y)$$

in the sense that $\langle \chi, G(\cdot, x) \rangle = \chi(x)$ for all $\chi \in \mathbb{P}_n(\mathbb{S}^d)$.

For hyperinterpolation w/o quadrature exactness:

$$\mathcal{L}_n f(x) = \sum_{\ell=0}^{n} \sum_{k=1}^{Z(d,\ell)} \left(\sum_{j=1}^{m} w_j f(x_j) Y_{\ell,k}(x_j) \right) Y_{\ell,k}(x) = \sum_{j=1}^{m} w_j f(x_j) G_n(x, x_j)$$

$$\langle \mathcal{L}_n \chi, \chi \rangle = \left\langle \sum_{j=1}^{m} w_j \chi(x_j) G_n(x, x_j), \chi(x) \right\rangle = \sum_{j=1}^{m} w_j \chi(x_j)^2$$

Theorem (An and W. '24 JoC)

Assume the quadrature rule **satisfies the MZ property**. Then for any $f \in C(\Omega)$:

- $\|\mathcal{L}_n f\|_{L^2} \leq \sqrt{1+\eta}\left(\sum_{j=1}^{m} w_j\right)^{1/2} \|f\|_\infty$;
- $$\|\mathcal{L}_n f - f\|_{L^2} \leq \left(\sqrt{1+\eta}\left(\sum_{j=1}^{m} w_j\right)^{1/2} + |\mathbb{S}^d|^{1/2}\right) E_n(f) + \sqrt{\eta^2+4\eta}\,\|\chi^*\|_{L^2},$$

where $\mathcal{L}_n$ stands for hyperinterpolation without quadrature exactness.

Note: If the quadrature rule has exactness degree at least **1**, then

$$\sum_{j=1}^{m} w_j = \int_{\mathbb{S}^d} 1 \mathrm{d}\omega_d = |\mathbb{S}^d|.$$

Error bound investigated numerically

- The error bound is controlled by n and m
- Le Gia & Mhaskar (random points)
  $\Rightarrow \eta$ has a lower bound order $\sqrt{n^2 \log n / m}$
  $\Rightarrow \sqrt{\eta^2 + 4\eta}\|\chi^*\|_{L^2}$ has a lower bound of order $m^{-1/4}$ w.r.t. m, and it increases as n enlarges

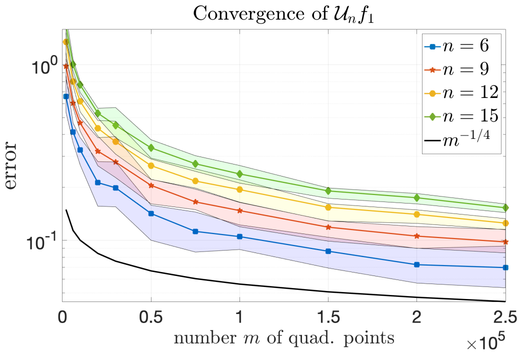


Figure: Approximating $f_1(x) = (x_1 + x_2 + x_3)^2 \in \mathbb{P}_6(\mathbb{S}^2)$.

- $f_2(x_1, x_2, x_3) := |x_1 + x_2 + x_3| + \sin^2(1 + |x_1 + x_2 + x_3|)$
- The Franke function for the sphere

$$
\begin{aligned}
f_3(x_1, x_2, x_3) :=& 0.75 \exp(-((9x_1 - 2)^2)/4 - ((9x_2 - 2)^2)/4 - ((9x_3 - 2)^2)/4) \\
&+ 0.75 \exp(-((9x_1 + 1)^2)/49 - ((9x_2 + 1))/10 - ((9x_3 + 1))/10) \\
&+ 0.5 \exp(-((9x_1 - 7)^2)/4 - ((9x_2 - 3)^2)/4 - ((9x_3 - 5)^2)/4) \\
&- 0.2 \exp(-((9x_1 - 4)^2) - ((9x_2 - 7)^2) - ((9x_3 - 5)^2)) \in C^\infty(\mathbb{S}^2)
\end{aligned}
$$

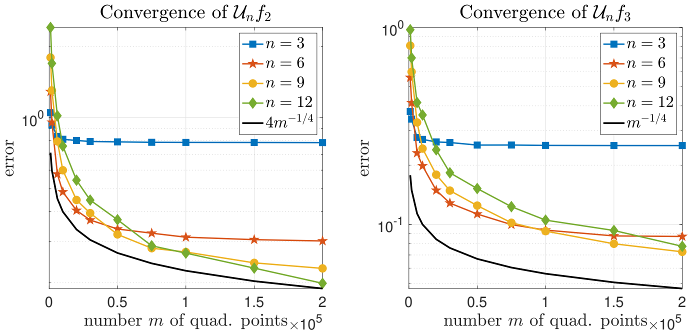


Figure: Approximating f_2 and f_3 (the notation $\mathcal{U}_n$ stands for hyperinterpolation, as adopted in our publication).

Applications to Fredholm integral equations of the second kind

☞ Consider the Fredholm integral equation of the second kind

$$\boxed{\varphi(\boldsymbol{x}) - \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})d\omega(\boldsymbol{y}) = f(\boldsymbol{x})}$$

on $\mathbb{S}^2$, where $|\boldsymbol{x}-\boldsymbol{y}| := \sqrt{2(1-\boldsymbol{x}\cdot\boldsymbol{y})}$ denotes the Euclidean distance between points $\boldsymbol{x}$ and $\boldsymbol{y}$ on $\mathbb{S}^2$.

- The inhomogeneous term f, the kernel K, and the solution φ are assumed to be **continuous**.
- The weight function $h : (0,\infty) \to \mathbb{R}$ is allowed to be **weakly singular**, i.e., h is continuous for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^2$ with $\boldsymbol{x} \neq \boldsymbol{y}$, and there exists positive constants M and $\alpha \in (0,2]$ such that

  $$|h(|\boldsymbol{x}-\boldsymbol{y}|)| \leq M|\boldsymbol{x}-\boldsymbol{y}|^{\alpha-2};$$

  to be strengthened to $|h(|\boldsymbol{x}-\boldsymbol{y}|)| \leq M|\boldsymbol{x}-\boldsymbol{y}|^{(\alpha-2)/2}$ for analysis.
- It is assumed that the homogeneous equation has no non-trivial solution; then **classic Riesz theory** $\Rightarrow$ the inhomogeneous equation has a unique solution continuously depending on f.

Singular kernel, modified moments, and semi-analytical approach

☞ Numerically evaluating singular integrals is **risky**: as quadrature points approach the singularity, the scheme becomes increasingly unstable.

☞ **Funk–Hecke formula:** Let $g \in L^1(-1,1)$ and $\boldsymbol{x} \in \mathbb{S}^2$. Then

$$\int_{\mathbb{S}^2} g(\boldsymbol{x}\cdot\boldsymbol{y}) Y_{\ell,k}(\boldsymbol{y}) d\omega(\boldsymbol{y}) = \mu_\ell Y_{\ell,k}(\boldsymbol{x}),$$

where

$$\mu_\ell := 2\pi \int_{-1}^{1} g(2^{1/2}(1-t)^{1/2}) P_\ell(t) dt,$$

and $P_\ell(t)$ is the standard Legendre polynomial of degree ℓ.

☞ **Modified moments:** Computing the singular part analytically

$$\boxed{\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|) Y_{\ell,k}(\boldsymbol{y}) d\omega(\boldsymbol{y}) = \mu_\ell Y_{\ell,k}(\boldsymbol{x}),}$$

where $|\boldsymbol{x}-\boldsymbol{y}| := \sqrt{2(1-\boldsymbol{x}\cdot\boldsymbol{y})}$ and

$$\boxed{\mu_\ell := 2\pi \int_{-1}^{1} h(2^{1/2}(1-t)^{1/2}) P_\ell(t) dt.}$$

☞ **Example 1:** $h(|\mathbf{x}-\mathbf{y}|) = |\mathbf{x}-\mathbf{y}|^{\nu}$ with $-2 < \nu < 0$. Then

$$\boxed{\mu_\ell = 2^{\nu+2}\pi\left(-\frac{\nu}{2}\right)_\ell \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\ell+\frac{\nu}{2}+2\right)}} \text{ with } (x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} \text{ for } x > 0.$$

☞ **Example 2:** $h(|\mathbf{x}-\mathbf{y}|) = \log|\mathbf{x}-\mathbf{y}|$. Then

$$\boxed{\mu_\ell = 2\pi\int_{-1}^{1}\log(2^{1/2}(1-t)^{1/2})P_\ell(t)dt} = \pi\int_{-1}^{1}\log(2(1-t))P_\ell(t)dt.$$

Numerically,

$$\mu_0 = \pi\int_{-1}^{1}\log(2(1-t))dt = \pi(4\log 2 - 2),$$

$$\mu_\ell = -\pi\int_{-1}^{1}\sum_{k=1}^{\infty}\frac{t^k}{k}P_\ell(t)dt = -\pi\sum_{k=1}^{\ell}\frac{1}{k}\int_{-1}^{1}t^kP_\ell(t)dt, \quad \ell = 1, 2, \ldots.$$

☞ **Example 3:** $h(|\mathbf{x}-\mathbf{y}|) = |\mathbf{x}-\mathbf{y}|^{\nu_1}|\mathbf{x}+\mathbf{y}|^{\nu_2}$ with $-2 \le \nu_1, \nu_2 < 0$. Then

$$\boxed{\mu_\ell = 2^{(\nu_1+\nu_2)/2}(2\pi)\int_{-1}^{1}(1-t)^{\nu_1/2}(1+t)^{\nu_2/2}P_\ell(t)dt.}$$

Quadrature rule for the integral operator

For the integral operator $\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})d\omega(\boldsymbol{y})$, we approximate it as

$$\begin{aligned}
&\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)\mathcal{L}_n\left(K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})\right)d\omega(\boldsymbol{y})\\
&=\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)\left(\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\langle K(\boldsymbol{x},\cdot)\varphi, Y_{\ell,k}\rangle_m Y_{\ell,k}(\boldsymbol{y})\right)d\omega(\boldsymbol{y})\\
&=\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\left(\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y})\right)\langle K(\boldsymbol{x},\cdot)\varphi, Y_{\ell,k}\rangle_m\\
&=\sum_{j=1}^{m} w_j\left(\sum_{\ell=0}^{n}\sum_{k=1}^{2\ell+1}\left(\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)Y_{\ell,k}(\boldsymbol{y})d\omega(\boldsymbol{y})\right)Y_{\ell,k}(\boldsymbol{x}_j)\right)K(\boldsymbol{x},\boldsymbol{x}_j)\varphi(\boldsymbol{x}_j)\\
&=:\sum_{j=1}^{m} W_j(\boldsymbol{x})K(\boldsymbol{x},\boldsymbol{x}_j)\varphi(\boldsymbol{x}_j),
\end{aligned}$$

Two-stage numerical scheme for the integral equation

Let φ_γ denotes the numerical solution, where the triple index $\gamma := (m, n, \eta)$ encodes the Marcinkiewicz–Zygmund property.

$$\varphi_\gamma(\boldsymbol{x}) - \sum_{j=1}^{m} W_j(\boldsymbol{x}) K(\boldsymbol{x}, \boldsymbol{x}_j) \varphi_\gamma(\boldsymbol{x}_j) = f(\boldsymbol{x}), \quad i = 1, \ldots, m$$

☞ **First stage** We set $\boldsymbol{x} = \boldsymbol{x}_j$, $j = 1, \ldots, m$, then numerically solves the obtained system of linear equations

$$\varphi_\gamma(\boldsymbol{x}_i) - \sum_{j=1}^{m} W_j(\boldsymbol{x}_i) K(\boldsymbol{x}_i, \boldsymbol{x}_j) \varphi_\gamma(\boldsymbol{x}_j) = f(\boldsymbol{x}_i), \quad i = 1, \ldots, m$$

for the quantities $\varphi_\gamma(\boldsymbol{x}_j)$, $j = 1, \ldots, m$.

☞ **Second stage** The value of $\varphi_\gamma(\boldsymbol{t})$ at any $\boldsymbol{t} \in \mathbb{S}^2$ can be evaluated by the direct usage of

$$\varphi_\gamma(\boldsymbol{t}) = f(\boldsymbol{t}) + \sum_{j=1}^{m} W_j(\boldsymbol{t}) K(\boldsymbol{t}, \boldsymbol{x}_j) \varphi_\gamma(\boldsymbol{x}_j).$$

Numerical analysis for the numerical scheme

Let

$$(A\varphi)(\boldsymbol{x}) := \int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|)K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})d\omega(\boldsymbol{y})$$

$$(A_\gamma\varphi)(\boldsymbol{x}) := \sum_{j=1}^{m} W_j(\boldsymbol{x})K(\boldsymbol{x},\boldsymbol{x}_j)\varphi(\boldsymbol{x}_j)$$

☞ For

$$\varphi - A\varphi = f,$$

Riesz theory ⇒ $(I-A)^{-1} : C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ exists and is bounded ✓

☞ For

$$\varphi_\gamma - A_\gamma\varphi_\gamma = f \quad \text{and} \quad (I-A_\gamma)(\varphi_\gamma-\varphi) = (A_\gamma - A)\varphi \ \ ?$$

The existence of φ_γ and the error bound of $\|\varphi_\gamma - \varphi\|_{L^\infty}$ depend on the **existence and boundedness of $(I-A_\gamma)^{-1}$ (?)**.

The identity $(I-A)^{-1} = I + (I-A)^{-1}A$ suggests that

$$B_\gamma := I + (I-A)^{-1}A_\gamma$$

is an approximate inverse for $I - A_\gamma$. Note that

$$(I-A)B_\gamma(I-A_\gamma) = \cdots = (I-A) - (A_\gamma - A)A_\gamma,$$

which is equivalent to

$$B_\gamma(I-A_\gamma) = I - S_\gamma,$$

where $S_\gamma := (I-A)^{-1}(A_\gamma - A)A_\gamma$. If we **assume**

$$\|(I-A)^{-1}(A_\gamma - A)A_\gamma\| < 1,$$

Neumann series $\Rightarrow$ $(I-S_\gamma)^{-1}$ exists and is bounded by

$$\|(I-S_\gamma)^{-1}\| \leq \frac{1}{1-\|S_\gamma\|}.$$

Then $I - A_\gamma$ is an injection. If we **assume A_γ is compact**, then Fredholm alternative $\Rightarrow$ $(I-A_\gamma)^{-1}$ exists and

$$(I-A_\gamma)^{-1} = (I-S_\gamma)^{-1}B_\gamma.$$

Key Lemma

Assume that operators A_γ are compact and $\gamma \in \Gamma$ such that the sequence $\{A_\gamma\}$ satisfies

$$\|(I - A)^{-1}(A_\gamma - A)A_\gamma\| < 1,$$

Then the inverse operators $(I - A_\gamma)^{-1} : C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ exist and are bounded by

$$\|(I - A_\gamma)^{-1}\| \leq \frac{1 + \|(I - A)^{-1}A_\gamma\|}{1 - \|(I - A)^{-1}(A_\gamma - A)A_\gamma\|}.$$

For solutions of the equations

$$\varphi - A\varphi = f \quad \text{and} \quad \varphi_\gamma - A_\gamma\varphi_\gamma = f,$$

we have the error estimate

$$\|\varphi_\gamma - \varphi\|_{L^\infty} \leq \frac{1 + \|(I - A)^{-1}A_\gamma\|}{1 - \|(I - A)^{-1}(A_\gamma - A)A_\gamma\|}\|(A_\gamma - A)\varphi\|_{L^\infty}.$$

- Applying previous approximation results of hyperinterpolation, we can verify A_γ is compact and $\|(I-A)^{-1}(A_\gamma - A)A_\gamma\| < 1$.
- We need $h(2^{1/2}(1-t)^{1/2}) \in L^1(-1,1) \cap L^2(-1,1)$.

Theorem (An & W. '24)

Let $\gamma = (m, n, \eta) \in \Gamma$ with sufficiently large n and sufficiently small η. Then

$$\|\varphi_\gamma\|_{L^\infty} \leq C_1(m,n,\eta)\|f\|_{L^\infty},$$

where $C_1(m,n,\eta) > 0$ is some constant decreasing as n grows or η decreases. Moreover, there exists $\boldsymbol{x}_0 \in \mathbb{S}^2$ such that

$$\|\varphi_\gamma - \varphi\|_{L^\infty} \leq C_2(m,n,\eta)\left(E_n(K(\boldsymbol{x}_0,\cdot)\varphi) + \sqrt{\eta^2 + 4\eta}\|\chi^*\|_{L^2}\right),$$

where $C_2(m,n,\eta) > 0$ is some constant decreasing as n grows or η decreases, and χ^* stands for the best approximation polynomial of $K(\boldsymbol{x}_0,\cdot)\varphi(\cdot)$ in $\mathbb{P}_n$.

Numerical experiments for the integral equation solver

☞ **Toy example setting:** For various singular kernel h and continuous kernel K, let $\varphi \equiv 1 \Rightarrow f = ? \Rightarrow$ Solve for φ_γ and compare with 1.

☞ **Point sets** $\{\boldsymbol{x}_j\}_{j=1}^m$ for the first stage: We investigate four kinds of point distributions on sphere:

- Spherical t-designs;
- Minimal energy points: a set of points $\{x_j\}_{j=1}^m \subset \mathbb{S}^2$ that minimizes the Coulomb energy
$$\sum_{i,j=1}^m \frac{1}{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|_2};$$
- Fekete points: a set of points that maximizes the determinant $\det\left(p_i\left(x_j\right)\right)_{i,j=1}^{d_n}$ for polynomial interpolation.
- Equal area points.

☞ **Validation points** for the second stage: 5,000 uniformly distributed points on $\mathbb{S}^2$.

Example 1: $h \equiv 1$ with

$$W_j(\boldsymbol{x}) = w_j \left(\sum_{\ell=0}^{n} \sum_{k=1}^{2\ell+1} \left(\int_{\mathbb{S}^2} h(|\boldsymbol{x}-\boldsymbol{y}|) Y_{\ell,k}(\boldsymbol{y}) d\omega(\boldsymbol{y}) \right) Y_{\ell,k}(\boldsymbol{x}_j) \right) \equiv w_j.$$

We let $K(\boldsymbol{x}, \boldsymbol{y}) = \sin(10|\boldsymbol{x}-\boldsymbol{y}|)$, thus

$$f(\boldsymbol{x}, \boldsymbol{y}) = 1 - 2\pi \int_{-1}^{1} \sin\left(10\sqrt{2(1-t)}\right) dt \approx 1.455449001125579.$$

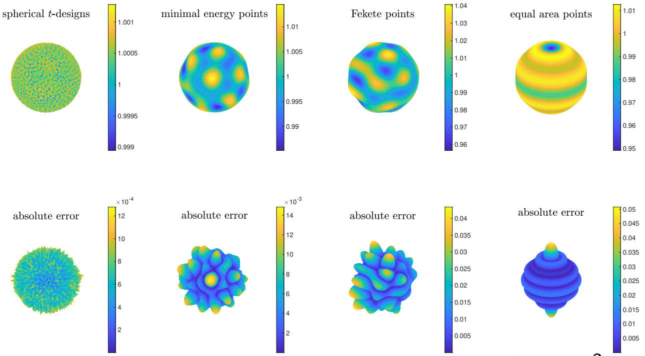


Figure: Numerical solutions with $n = 20$ and $m = (2n+1)^2$.

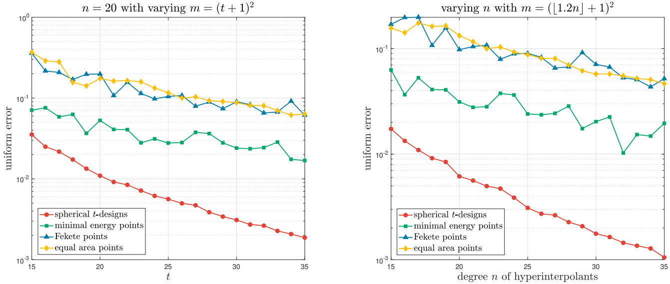


Figure: Non-singular $h = 1$ and oscillatory $K(\boldsymbol{x}, \boldsymbol{y}) = \sin(10|\boldsymbol{x} - \boldsymbol{y}|)$: Uniform errors with different n and m.

Example 2: $h(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|^{-0.5}$ with modified moments (and hence $W_j(\boldsymbol{x})$) analytically evaluated. We let $K(\boldsymbol{x},\boldsymbol{y}) = \cos(10|\boldsymbol{x}-\boldsymbol{y}|)$, thus

$$f(\boldsymbol{x},\boldsymbol{y}) \equiv 1 - 2\pi \int_{-1}^{1} \left(\sqrt{2(1-t)}\right)^{-0.5} \cos\left(10\sqrt{2(1-t)}\right) dt$$
$$\approx 0.303738699125466.$$

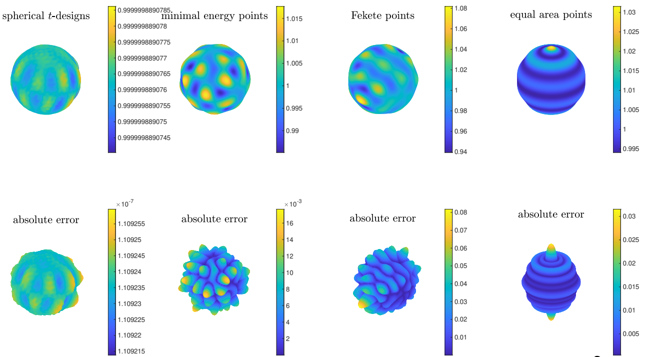


Figure: Numerical solutions with $n = 20$ and $m = (2n+1)^2$.

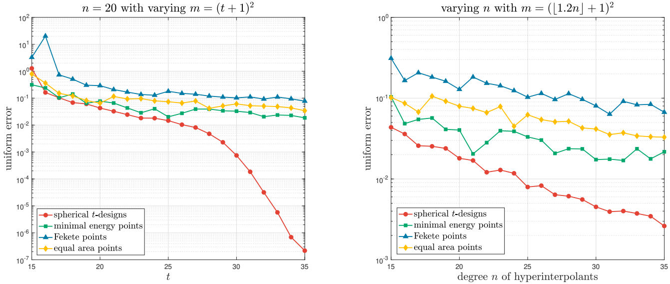


Figure: Singular $h(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|^{-0.5}$ and oscillatory $K(\boldsymbol{x}, \boldsymbol{y}) = \cos(10|\boldsymbol{x} - \boldsymbol{y}|)$: Uniform errors with different n and m.

Example 3: $h(\boldsymbol{x}, \boldsymbol{y}) = \log|\boldsymbol{x} - \boldsymbol{y}|$ with modified moments (and hence $W_j(\boldsymbol{x})$) stably evaluated. We let $K = 1$ and

$$f(\boldsymbol{x}, \boldsymbol{y}) \equiv 1 - 2\pi \int_{-1}^{1} \log\left(\sqrt{2(1-t)}\right) dt = 1 - \pi(4\log 2 - 2).$$

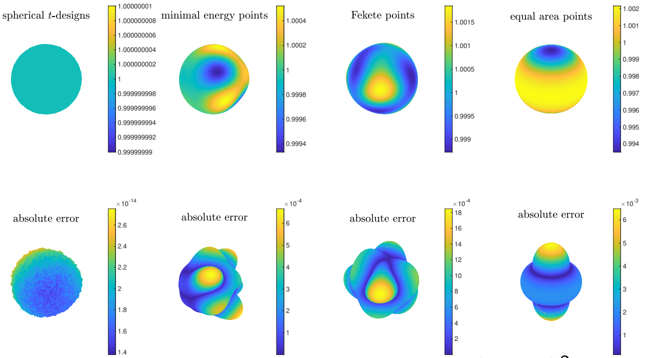


Figure: Numerical solutions $n = 5$ and $m = (2n+1)^2$.

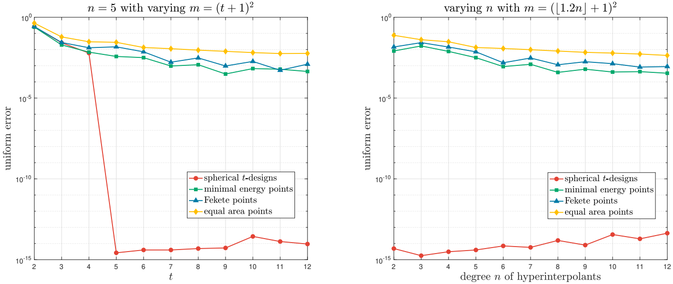


Figure: Singular $h(\boldsymbol{x}, \boldsymbol{y}) = \log |\boldsymbol{x} - \boldsymbol{y}|$ and non-oscillatory $K = 1$: Uniform errors with different n and m.

Example 4: $h(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|^{-0.5}|\boldsymbol{x}+\boldsymbol{y}|^{-0.5}$ with modified moments (and hence $W_j(\boldsymbol{x})$) stably evaluated. We let $K(\boldsymbol{x},\boldsymbol{y}) = \sin(10|\boldsymbol{x}-\boldsymbol{y}|)$ and

$$f(\boldsymbol{x},\boldsymbol{y}) = 1 - 2\pi \int_{-1}^{1} \left(\sqrt{2(1-t)}\sqrt{2(1+t)}\right)^{-0.5} \sin\left(10\sqrt{2(1-t)}\right) dt$$
$$\approx 0.011007492841040.$$

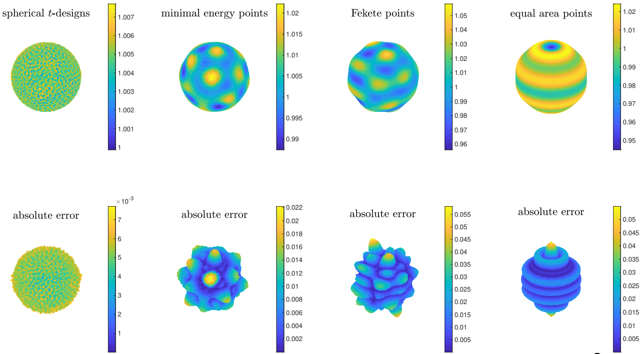


Figure: Numerical solutions with $n = 20$ and $m = (2n+1)^2$.

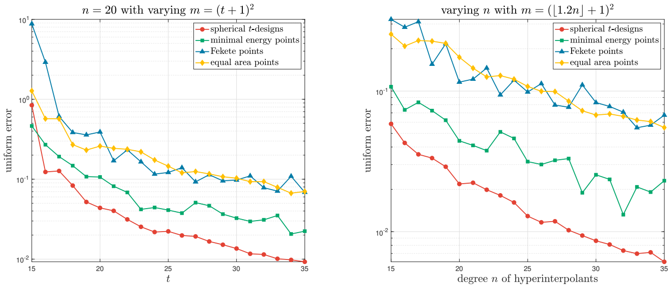


Figure: Singular $h(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|^{-0.5}|\boldsymbol{x} + \boldsymbol{y}|^{-0.5}$ and oscillatory $K(\boldsymbol{x}, \boldsymbol{y}) = \sin(10|\boldsymbol{x} - \boldsymbol{y}|)$: Uniform errors with different n and m.

Thanks for your attention.

Photo taken from Grass Island/Tap Mun, Hong Kong.